CHARACTERIZATION OF DISTRIBUTIONS THROUGH CONDITIONAL EXPECTATION OF ORDERED STATISTICS

THESIS

SUBMITTED FOR THE AWARD OF THE DEGREE OF

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BY

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Dedicated to My Parents and
My beloved Sister Arzoo Khan

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Certified that the pre-submission presentation of the Ph. D. (Statistics) thesis entitled "**Characterization of Distributions through Conditional Expectation of Ordered Statistics**" was held on 02.03.2013 at 11:00 a.m. in the Department by **Mr. Md. Izhar Khan**. He presented his work satisfactorily.

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Certified that the material contained in this thesis entitled **"Characterization of Distributions through Conditional Expectation of Ordered Statistics"**, submitted by **Mr. Md. Izhar Khan** for the award of the degree of Doctor of Philosophy in Statistics is original.

This work was carried out under my supervision. In my opinion the work is sufficient for consideration for the award of the Ph. D. degree in Statistics.

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md Izhar Khan

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CONTENTS

PREFACE

This thesis entitled **"Characterization of Distributions through Conditional Expectation of Ordered Statistics"** is based on five chapters, in which Chapter I is introductory in nature and deals with the basic concepts and results needed in the subsequent chapters.

Chapter II deals with the characterization of a general form of continuous distributions. These characterization results are based on contrast of conditional expectations of order statistics, record values and generalized order statistics conditioned on non-adjacent order statistic, record and generalized order statistic, respectively.

Chapter III embodies the result based on characterization of continuous distributions by considering conditional expectation of generalized order statistics. Further, using Meijer's G function, conditional expectation of generalized order statistics based on two non- adjacent generalized order statistics has been used to characterize a general form of distribution.

Chapter IV is based on characterization of a generalized family of continuous distributions through conditional expectation of dual generalized order statistic, conditioned on a non-adjacent dual generalized order statistic.

Chapter V deals with exact expressions for single and product moments of record statistics for two parameters Burr type XII distribution. The means, variances and covariances of the record statistics are computed for various values of the shape parameters. These values are then used to compute the coefficients of the best linear unbiased estimators of the location and scale parameters. The variances of these estimators are also obtained. The predictors of the future record statistics are discussed as well.

In the end, a comprehensive bibliography is given which have been referred by us and are relevant to our work.

CHAPTER-I

PRELIMINARIES AND BASIC CONCEPTS

We have introduced here concepts and results which may be needed in the grasping of the results in the subsequent Chapters.

1. Order statistics [David and Nagaraja, 2003]:

Let $X_1, X_2, \ldots, X_n$ be n random variables, if random variables $X_1, X_2, \ldots, X_n$ are arranged in ascending order of magnitude such that $X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$, then $X_{r:n}$ is called the r^{th} order statistic. $X_{1:n}$ = min $[X_1, X_2, \ldots, X_n]$ and $X_{n:n}$ = max $[X_1, X_2, \ldots, X_n]$ are called extreme order statistics or the smallest and the largest order statistics, respectively. The subject of order statistics deals with the properties and applications of these ordered random variables and of functions involving them (David and Nagaraja, 2003). Asymptotic theory of extremes and related developments of order statistics are well described in an applausive work of Galambos (1987). Also, references may be made to Sarhan and Greenberg (1962), Balakrishnan and Cohen (1991), Arnold *et al.* (1992) and the references therein. It is different from the rank order statistics in which the order of the value of observation rather than its magnitude is considered. It plays an important role both in the model building and in the statistical inference.

For example, extreme values are important in oceanography (waves and tides), material strength (strength of a chain depends on the weakest link) and meteorology (extremes of temperature, pressure etc). Another very interesting application of the order statistics is found in reliability theory. The r^{th} order statistic $X_{r:n}$ in a sample of size n represents the lifelength of a $(n-r+1)$-out- of - n- system. This system consists of n components of the same kind with independently distributed life lengths. All n components start working simultaneously and the system fails, if r or more component fails. In other words, $n-r+1$ components are necessary for the system to work. For $r=1$, the system corresponds to a series system whereas for $r=n$, it corresponds to a parallel system.

2. Distribution of order statistics [David and Nagaraja, 2003]:

Here in this section, we will discuss the basic distribution theory of order statistics for continuous population.

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a continuous population with probability density function (*pdf*) $f(x)$ and the distribution function (*df*) $F(x)$. Then the *pdf* of $X_{r:n}, 1 \le r \le n$, the r^{th} order statistic is given by

$$f_{r:n}(x) = c_{r:n}[F(x)]^{r-1}[1-F(x)]^{n-r} f(x), \; -\infty < x < \infty \tag{2.1}$$

where

$$c_{r:n} = \frac{n!}{(r-1)!(n-r)!} = [B(r, n-r+1)]^{-1} \tag{2.2}$$

In particular, the *pdf* of the smallest and the largest order statistics are

$$f_{1:n}(x) = n[1-F(x)]^{n-1} f(x), \quad -\infty < x < \infty \tag{2.3}$$

$$f_{n:n}(x) = n[F(x)]^{n-1} f(x), \quad -\infty < x < \infty \tag{2.4}$$

and the *df* of $X_{r:n}$ is

$$F_{r:n}(x) = P[X_{r:n} \le x]$$

$= P$ (at least r of $X_1, X_2, \ldots, X_n$ are less than or equal to x)

$$= \sum_{i=r}^{n} P(\text{exactly } i \text{ of } X_1, X_2, \dots, X_n \text{ are less than or equal to } x)$$

$$= \sum_{i=r}^{n} \binom{n}{i} [F(x)]^i [1 - F(x)]^{n-i} \tag{2.5}$$

$$= c_{r:n} \int_0^{F(x)} t^{r-1} (1-t)^{n-r} dt \tag{2.6}$$

$$= \mathrm{I}_{F(x)}(r, n-r+1) \tag{2.7}$$

where

$$I_p(a,b) = \frac{1}{B(a,b)} \int_0^p t^{a-1}(1-t)^{b-1} dt$$

and $B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1} dt$

RHS of (2.7) is obtained by the relationship between binomial sums and incomplete beta function. It may also be expressed in negative binomial sums as (Khan, 1991).

$$F_{r:n}(x) = \sum_{i=0}^{n-r} \binom{i+r-1}{r-1} [F(x)]^r [1 - F(x)]^i, \; -\infty < x < \infty \tag{2.8}$$

The k^{th} moment of $X_{r:n}$ is given by

$$\alpha_{r:n}^{(k)} = E[X_{r:n}^k] = \int_{-\infty}^{\infty} x^k f_{r:n}(x) dx \tag{2.9}$$

The joint *pdf* of $X_{r:n}$ and $X_{s:n}$, $1 \le r < s \le n$, is given by

$$f_{r,s:n}(x,y) = c_{r,s:n} [F(x)]^{r-1} [F(y) - F(x)]^{s-r-1}$$

$$\times [1 - F(y)]^{n-s} f(x) f(y), \; -\infty < x < y < \infty \tag{2.10}$$

where

$$c_{r,s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!} = [B(r, s-r, n-s+1)]^{-1} \tag{2.11}$$

The joint df is

$$F_{r,s:n}(x,y) = P(X_{r:n} \le x, X_{s:n} \le y)$$

$$= P(\text{at least } r \text{ of } X_1, X_2, ..., X_n \text{ are at most } x$$

$$\text{and at least } s \text{ of } X_1, X_2, ..., X_n \text{ are at most } y)$$

$$= \sum_{j=s}^{n} \sum_{i=r}^{j} P(\text{exactly } i \text{ of } X_1, X_2, ..., X_n \text{ are at most } x$$

$$\text{and exactly } j \text{ of } X_1, X_2, ..., X_n \text{ are at most } y)$$

$$= \sum_{j=s}^{n} \sum_{i=r}^{j} \frac{n!}{i!(j-i)!(n-j)!}[F(x)]^i$$

$$\times [F(y) - F(x)]^{j-i}[1 - F(y)]^{n-j} \tag{2.12}$$

We can write the joint df of $X_{r:n}$ and $X_{s:n}$ in (2.12) equivalently as:

$$F_{r,s:n}(x,y) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!} \int_0^{F(x)} \int_u^{F(y)} u^{r-1}(v-u)^{s-r-1}$$

$$\times (1-v)^{n-s}\, dv\, du = I_{F(x),F(y)}(r, s-r, n-s+1),$$

$$-\infty < x < y < \infty \tag{2.13}$$

which is incomplete bivariate beta function.

It may be noted that for $x \ge y$, the inequality $X_{s:n} \le y$ implies $X_{r:n} \le x$, so that

$$F_{r,s:n}(x,y) = F_{s:n}(y) \tag{2.14}$$

The product moments of $X_{r:n}$ and $X_{s:n}$ is given by

$$\alpha_{r,s:n}^{(j,k)} = E[X_{r:n}^{j} X_{s:n}^{k}]$$

$$= \iint\limits_{-\infty<x<y<\infty} x^j y^k f_{r,s:n}(x,y)\,dx\,dy \tag{2.15}$$

Remarks:

The ranking of random variables $X_1, X_2, ..., X_n$ is preserved under any monotonic increasing transformation of the random variables.

1. Regarding the probability integral transformation, if $X_{r:n}$, $1 \le r \le n$, be the order statistic from a continuous *df* $F(x)$, then the transformation $U_{r:n} = F(X_{r:n})$ produces a random variable which is the r^{th} order statistic from a uniform distribution on $U(0, 1)$.

2. Even if $X_1, X_2, ..., X_n$ are independent random variables, order statistics are not independent random variables.

3. Let $X_1, X_2, ..., X_n$ be *iid* random variables from a continuous distribution, then the set of order statistics $\{X_{1:n}, X_{2:n}, ..., X_{n:n}\}$ is both sufficient and complete (Lehmann, 1986).

4. Let X be a continuous random variable with $E(X_{r:n}) = \alpha_{r:n}$,

   (a) If $\alpha = E(X)$ exits then $\alpha_{r:n}$ exists, but converse is not necessarily true. That is, $\alpha_{r:n}$ may exist for certain (but not all) values of r, even though α may not exist.

   (b) $\alpha_{r:n}$ for all n determine the distribution completely.

3. Conditional distribution of order statistics [David and Nagaraja, 2003]:

Let $X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$ be the order statistics from a continuous random variable having the *pdf* $f(x)$ and the *df* $F(x)$.

Then the conditional *pdf* of $X_{s:n} = y$ given $X_{r:n} = x$, $1 \le r < s \le n$, is

$$\left[\frac{(n-r)!}{(s-r-1)!(n-s)!}\right] \frac{[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s}}{[1-F(x)]^{n-r}} f(y) \text{ , } x \le y \quad (3.1)$$

$$= c_{s-r:n-r}\left[\frac{F(y)-F(x)}{1-F(x)}\right]^{s-r-1}\left[1-\frac{F(y)-F(x)}{1-F(x)}\right]^{n-s}\frac{f(y)}{1-F(x)}$$

which is just the unconditional *pdf* of $(s-r)^{th}$ order statistic in a sample of size $(n-r)$ drawn from $\dfrac{f(y)}{1-F(x)}$, $y \geq x$, that is from the parent distribution truncated on the left at x.

Similarly, the conditional *pdf* of $X_{r:n}=x$ given $X_{s:n}=y, 1 \leq r < s \leq n$ is

$$\frac{(s-1)!}{(r-1)!(s-r-1)!}\frac{[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}}{[F(y)]^{s-1}}f(x) \ , \ x \leq y \quad (3.2)$$

$$= c_{r:s-1}\left[\frac{F(x)}{F(y)}\right]^{r-1}\left[1-\frac{F(x)}{F(y)}\right]^{s-r-1}\frac{f(x)}{F(y)}$$

which is the unconditional *pdf* of $X_{r:s-1}$ truncated at y on the right.

Remark: Order statistics in a sample from a continuous distribution form a Markov chain, that is

$$f(X_{k:n} \mid X_{1:n}=x_1,\ldots,X_{r:n}=x_r,\ldots X_{s:n}=x_s,\ldots X_{n:n}=x_n)$$
$$= f(X_{k:n} \mid X_{r:n}=x_r,\, X_{s:n}=x_s)$$

Therefore, because of the Markovian properties of order statistics, it is of no use to condition it on more than two order statistics.

4. Record values and record times [Ahsanullah, 1995]:

It is difficult to separate the theory of records from the theory of order statistics. Records are closely related to the extremal order statistics. Suppose that $X_1, X_2, \ldots$ is a sequence of independent and identically distributed random variables with the *df* $F(x)$. Let $Y_j = \max[X_1, X_2, \ldots, X_j]$ for $j \geq 1$. We say X_j is an upper record if

$Y_j > Y_{j-1}$, $j > 1$. An analogous definition deals with lower record values. One can transform the upper record by replacing the original sequence of (X_j) by $(-X_j,\ j \geq 1)$ or if $P(X_i > 0) = 1$ for all i then by $\left\{\frac{1}{X_i}, i \geq 1\right\}$, the lower record value of this sequence will correspond to the upper record values of the original sequence.

The indices at which upper record values occur are given by the record times $\{U(n)\}$, $n > 0$. That is $X_{U(n)}$ is the n^{th} upper record, where $U(n) = \min[j > U(n-1) : X_j > X_{U(n-1)},\ n > 1]$ and $U(1) = 1$. The distribution of $U(n)$, $n \geq 1$ does not depend on F. Further, we will denote $L(n)$ as the indices where the lower record values occur. By assumption $U(1) = L(1) = 1$. The distribution of $L(n)$ also does not depend on F.

Record values are observed in many situations of daily life as well as in many statistical applications. Often we are interested in observing new records and in recording them: e.g. Olympic records or world records in sports.

Record values are defined by Chandler (1952) as a model of successive extremes in a sequence of identically and independent random variables. It may also be helpful as a model for successively largest insurance claims in non-life insurance, for highest water-levels or highest temperatures. Record values are also useful in reliability theory.

5. Distribution of record values [Ahsanullah, 1995]:

Let $R(x)$ be a continuous function of x with $R(x) = -\ln \overline{F}(x)$, $0 < \overline{F}(x) < 1$ and $\overline{F}(x) = 1 - F(x)$. Here 'ln' is used for the natural logarithm.

If we define $F_{U(r)}(x)$ as the distribution function of $X_{U(r)}$ for $r \geq 1$, then we have ,

$$F_{U(r)}(x) = P[X_{U(r)} \leq x] \,. \tag{5.1}$$

The *pdf* of $X_{U(r)}$ is

$$f_{U(r)}(x) = \frac{R^{r-1}(x)}{(r-1)!} f(x), \ -\infty < x < \infty. \tag{5.2}$$

The joint *pdf* of $X_{U(r)}$ and $X_{U(s)}$ is

$$f_{U(r),U(s)}(x,y) = \frac{1}{(r-1)!(s-r-1)!} R^{r-1}(x)$$

$$\times [R(y) - R(x)]^{s-r-1} \frac{f(x)}{\overline{F}(x)} f(y), \quad -\infty < x < y < \infty. \tag{5.3}$$

Hence the conditional *pdf* of $X_{U(s)}$ given $X_{U(r)} = x$, is

$$f_{U(s)|U(r)}(y|x) = \frac{[R(y) - R(x)]^{s-r-1}}{(s-r-1)!} \frac{f(y)}{\overline{F}(x)},$$

$$-\infty < x < y < \infty. \tag{5.4}$$

The marginal *pdf* of the r^{th} lower record value can be obtained by using the *pdf* of the r^{th} upper record value by replacing $R(x)$ with $H(x) = -\ln F(x)$, $0 < F(x) < 1$

$$F_{L(r)}(x) = P[X_{L(r)} \leq x] \tag{5.5}$$

and the corresponding *pdf* $f_{L(r)}(x)$ can be written as

$$f_{L(r)}(x) = \frac{H^{r-1}(x)}{(r-1)!} f(x) \tag{5.6}$$

The joint *pdf* of $X_{L(r)}$ and $X_{L(s)}$ is

$$f_{L(r),L(s)}(x,y)=\frac{1}{(r-1)!(s-r-1)!}H^{r-1}(x)$$

$$\times[H(y)-H(x)]^{s-r-1}\frac{f(x)}{F(x)}f(y),\quad -\infty<y<x<\infty. \tag{5.7}$$

Hence the conditional *pdf* of $X_{L(s)}$ given $X_{L(r)}=x$, is

$$f_{L(s)|L(r)}(y\,|\,x)=\frac{[H(y)-H(x)]^{s-r-1}}{(s-r-1)!}\frac{f(y)}{F(x)},$$

$$-\infty<y<x<\infty. \tag{5.8}$$

6. Generalized order statistics [Kamps, 1995]:

Kamps introduced the model of generalized order statistics (gos) as follows:

Let $X_1,X_2,\ldots,X_n$ be a sequence of independent and identically distributed (iid) random variable (rv) with the *df* $F(x)$ and the *pdf* $f(x)$. Let $n\in N$, $n\geq 2$, $k>0$, $\widetilde{m}=(m_1,m_2,\ldots,m_{n-1})\in\Re^{n-1}$, $M_r=\sum_{j=r}^{n-1}m_j$, such that $\gamma_r=k+n-r+M_r>0$ for all $r\in\{1,2,\ldots,n-1\}$.

Then $X(r,n,\widetilde{m},k)$, $r=1,2,\ldots,n$ are called *gos* if their joint *pdf* is given by

$$k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}[1-F(x_i)]^{m_i}f(x_i)\right)[1-F(x_n)]^{k-1}f(x_n) \tag{6.1}$$

on the cone $F^{-1}(0+)<x_1\leq x_2\leq\cdots\leq x_n<F^{-1}(1)$ of $\Re^n$.

The joint density of the first r-*gos* is given by

$$f_{X(1,n,\tilde{m},k),\ldots,X(r,n,\tilde{m},k)}(x_1,x_2,\ldots,x_r)$$

$$=c_{r-1}\left(\prod_{i=1}^{r-1}[\overline{F}(x_i)]^{m_i} f(x_i)\right)[\overline{F}(x_r)]^{k+n-r+M_r-1} f(x_r) \tag{6.2}$$

on the cone $F^{-1}(0+) < x_1 \le x_2 \le \cdots \le x_n < F^{-1}(1)$.

Here we may consider two cases:

Case I: $m_1 = m_2 = \ldots = m_{n-1} = m$.

Case II: $\gamma_i \ne \gamma_j$, $i,j = 1,2,\ldots,n-1$, $i \ne j$.

For **Case I**, it is called $m-gos$ and the marginal density of the r^{th} $m-gos$ is given by [Kamps, 1995]:

$$f_{X(r,n,m,k)}(x) = \frac{c_{r-1}}{(r-1)!}[\overline{F}(x)]^{\gamma_r -1} f(x) g_m^{r-1}[F(x)] \tag{6.3}$$

and the joint *pdf* of $X(r,n,m,k)$ and $X(s,n,m,k)$, the r^{th} and s^{th} $m-gos$, $1 \le r < s \le n$, is

$$f_{X(r,n,m,k),X(s,n,m,k)}(x,y) = \frac{c_{s-1}}{(r-1)!(s-r-1)!}[\overline{F}(x)]^m g_m^{r-1}[F(x)]$$

$$\times[h_m(F(y)) - h_m(F(x))]^{s-r-1}[\overline{F}(y)]^{\gamma_s -1} f(x) f(y),$$

$$x < y. \tag{6.4}$$

The joint *pdf* of $X(r,n,m,k)$, $X(j,n,m,k)$ and $X(s,n,m,k)$, the r^{th}, j^{th} and s^{th} $m-gos$, $1 \le r < j < s \le n$, can similarly be given as

$$f_{X(r,n,m,k),X(j,n,m,k),X(s,n,m,k)}(x,t,y)$$

$$= c_{r,j,s:n}[\overline{F}(x)]^m g_m^{r-1}[F(x)][h_m(F(t)) - h_m(F(x)]^{j-r-1}$$

$$\times[h_m(F(y)) - h_m(F(t)]^{s-j-1}[\overline{F}(t)]^m[\overline{F}(y)]^{\gamma_s -1}$$

$$\times f(x)f(t)f(y),\ \alpha < x < t < y < \beta, \tag{6.5}$$

where

$$c_{r-1} = \prod_{i=1}^{r} \gamma_i , \quad \gamma_i = k + (n-i)(m+1) ,$$

$$h_m(x) = \begin{cases} -\dfrac{1}{m+1}(1-x)^{m+1}, & m \neq -1 \\ -\log(1-x) & , \quad m = -1 \end{cases}$$

and

$$g_m(x) = \int_0^x (1-t)^m \, dt = h_m(x) - h_m(0) , \; x \in [0,1) .$$

The conditional *pdf* of $X(s,n,m,k)$ given $X(r,n,m,k) = x$, $1 \le r < s \le n$, is given by

$$f_{X(s,n,m,k)|X(r,n,m,k)}(y \mid x)$$

$$= \frac{c_{s-1}}{(s-r-1)! c_{r-1}} \frac{[h_m(F(y)) - h_m(F(x))]^{s-r-1}}{[\overline{F}(x)]^{\gamma_r + 1}} [\overline{F}(y)]^{\gamma_s - 1} f(y),$$

$$x < y \qquad (6.6)$$

Now since $\lim\limits_{m \to -1} h_m(x) = \log\left(\dfrac{1}{1-x}\right)$, therefore, we will consider only the case $h_m(x) = -\dfrac{1}{m+1}(1-x)^{m+1}$ for all m, unless needed otherwise.

Thus (6.6) reduces to

$$f_{s|r}(y \mid x) = \frac{c_{s-1}}{c_{r-1}(s-r-1)!(m+1)^{s-r-1}} \left[1 - \left(\frac{\overline{F}(y)}{\overline{F}(x)} \right)^{m+1} \right]^{s-r-1}$$

$$\times \left[\frac{\overline{F}(y)}{\overline{F}(x)} \right]^{k+(m+1)(n-s)-1} \frac{f(y)}{\overline{F}(x)}, \quad x < y \qquad (6.7)$$

and the conditional distribution of $X(j,n,m,k)$ given $X(r,n,m,k)=x$ and $X(s,n,m,k)=y$, $1 \le r < j < s \le n$, is given by

$$f_{X(j,n,m,k)|X(r,n,m,k),X(s,n,m,k)}(t \mid x,y) = \frac{(s-r-1)!(m+1)}{(j-r-1)!(s-j-1)!}$$

$$\times \frac{[\{\overline{F}(x)\}^{m+1} - \{\overline{F}(t)\}^{m+1}]^{j-r-1}[\{\overline{F}(t)\}^{m+1} - \{\overline{F}(y)\}^{m+1}]^{s-j-1}}{[\{\overline{F}(x)\}^{m+1} - \{\overline{F}(y)\}^{m+1}]^{s-r-1}}$$

$$\times[\overline{F}(t)]^m f(t), \quad \alpha < x < t < y < \beta. \tag{6.8}$$

For **Case II**, the *pdf* of $X(r,n,\widetilde{m},k)$ is [Kamps and Cramer, 2001]:

$$f_{X(r,n,\widetilde{m},k)}(x) = c_{r-1} f(x) \sum_{i=1}^{r} a_i(r)[\overline{F}(x)]^{\gamma_i - 1} \tag{6.9}$$

and the joint *pdf* of $X(r,n,\widetilde{m},k)$ and $X(s,n,\widetilde{m},k)$, $1 \le r < s \le n$, is

$$f_{X(r,n,\widetilde{m},k),X(s,n,\widetilde{m},k)}(x,y) = c_{s-1} \sum_{i=r+1}^{s} a_i^{(r)}(s)\left(\frac{\overline{F}(y)}{\overline{F}(x)}\right)^{\gamma_i}$$

$$\times\left[\sum_{i=1}^{r} a_i(r)(\overline{F}(x))^{\gamma_i}\right]\frac{f(x)}{\overline{F}(x)}\frac{f(y)}{\overline{F}(y)}. \tag{6.10}$$

The joint *pdf* of $X(r,n,\widetilde{m},k)$, $X(j,n,\widetilde{m},k)$ and $X(s,n,\widetilde{m},k)$, $1 \le r < j < s \le n$, may similarly be given as

$$f_{X(r,n,\widetilde{m},k),X(j,n,\widetilde{m},k),X(s,n,\widetilde{m},k)}(x,t,y)$$

$$= c_{s-1}\left(\sum_{i=1}^{r} a_i(r)(\overline{F}(x))^{\gamma_i}\right)\left(\sum_{i=r+1}^{j} a_i^{(r)}(j)\left[\frac{\overline{F}(t)}{\overline{F}(x)}\right]^{\gamma_i}\right)$$

$$\times\left(\sum_{i=j+1}^{s} a_i^{(j)}(s)\left[\frac{\overline{F}(y)}{\overline{F}(t)}\right]^{\gamma_i}\right)\frac{f(x)}{\overline{F}(x)}\frac{f(t)}{\overline{F}(t)}\frac{f(y)}{\overline{F}(y)},$$

$$\alpha < x < t < y < \beta, \tag{6.11}$$

where

$$\gamma_i = k + n - i + M_i, \quad a_i(r) = \prod_{\substack{j=1 \\ j \neq i}}^{r} \frac{1}{(\gamma_j - \gamma_i)}, \; 1 \le i \le r \le n, \; i \neq j$$

and $$a_i^{(r)}(s) = \prod_{\substack{j=r+1 \\ j \neq i}}^{s} \frac{1}{(\gamma_j - \gamma_i)}, \; r+1 \le i \le s \le n, \; i \neq j.$$

Thus, the conditional *pdf* of $X(s,n,\widetilde{m},k)$ given $X(r,n,\widetilde{m},k) = x$, $1 \le r < s \le n$, is

$$f_{X(s,n,\widetilde{m},k)|X(r,n,\widetilde{m},k)}(y \mid x)$$

$$= \frac{c_{s-1}}{c_{r-1}} \sum_{i=r+1}^{s} a_i^{(r)}(s) \left[\frac{\overline{F}(y)}{\overline{F}(x)} \right]^{\gamma_i} \frac{f(y)}{[\overline{F}(y)]}, \qquad x \le y \tag{6.12}$$

and the conditional *pdf* of $X(j,n,\widetilde{m},k)$ given $X(r,n,\widetilde{m},k) = x$ and $X(s,n,\widetilde{m},k) = y, 1 \le r < j < s \le n$, is

$$f_{X(j,n,\widetilde{m},k)|X(r,n,\widetilde{m},k),X(s,n,\widetilde{m},k)}(t \mid x, y)$$

$$= \frac{\left[\sum_{i=r+1}^{j} a_i^{(r)}(j) \left\{ \frac{\overline{F}(t)}{\overline{F}(x)} \right\}^{\gamma_i} \right]}{\left[\sum_{i=r+1}^{s} a_i^{(r)}(s) \left\{ \frac{\overline{F}(y)}{\overline{F}(x)} \right\}^{\gamma_i} \right]} \left[\sum_{i=j+1}^{s} a_i^{(j)}(s) \left\{ \frac{\overline{F}(y)}{\overline{F}(t)} \right\}^{\gamma_i} \right] \frac{f(t)}{\overline{F}(t)}. \tag{6.13}$$

Generalized order statistics includes all the models related to ordered random variables. For instance it includes ordinary order statistics [David & Nagaraja(2003), Lawless (1982), Arnold *et al.* (1992), Balakrishnan and Rao (1998a, b)], order statistics with non integral sample size [Stigler (1977), Rohatgi and Saleh (1988)], records [Arnold *et al.* (1998), Nevzorov (2001)], k^{th} records [Dziubdziela and Kopocinski, 1976], sequential order statistics [Kamps (1995), Cramer and Kamps (2003)], Progressive Type-II right censoring [Balakrishnan and Aggrawala, 2000]. Choosing the parameters appropriately [Cramer, 2002], we get:

Table 1.1: Variants of the generalized order statistics

		$\gamma_n = k$	γ_r	m_r
i)	Sequential order statistics	α_n	$(n-r+1)\alpha_r$	$(\gamma_r - \gamma_{r+1} - 1)$
ii)	Ordinary order statistics	1	$n-r+1$	0
iii)	Record statistics	1	1	-1
iv)	Progressively type II censored order statistics	$R_n + 1$	$n-r+1+\sum_{j=r}^{n} R_j$	R_r
v)	Pfeifer's record statistics	β_n	β_r	$(\beta_r - \beta_{r+1} - 1)$

An alternative definition of *gos* is given by [Cramer and Kamps, 2003]:

$$X(r, n, \widetilde{m}, k) \stackrel{d}{=} F^{-1}\left(1 - \prod_{i=1}^{r} B_i\right), \tag{6.14}$$

where $B_1, \ldots, B_r$ are independent random variables with respective distribution functions Beta $(\gamma_j, 1), 1 \le j \le r$.

Let P_F stand for the probability measure on R determined by $F(x)$, then the *pdf* of $X(r,n,\widetilde{m},k)$ with respect to a measure P_F is given by [Cramer and Kamps, 2003]:

$$f_r(x) = c_{r-1}\, G_r(\overline{F}(x) \mid \gamma_1,\ldots,\gamma_r)\, I_{(\alpha,\beta)}(x), \tag{6.15}$$

where I_A denotes the indicator function and $G_r(x) = G_{r,r}^{r,0}(x \mid \gamma_1,\ldots,\gamma_r) = G_{r,r}^{r,0}\left(x \middle| \begin{matrix} \gamma_1,\ldots,\gamma_r \\ \gamma_1-1,\ldots,\gamma_r-1 \end{matrix}\right)$ is the particular Meijer's G-function defined by

$$G_{r,r}^{r,0}\left(s \middle| \begin{matrix} \gamma_1,\ldots,\gamma_r \\ \gamma_1-1,\ldots,\gamma_r-1 \end{matrix}\right) = \frac{1}{2\pi i}\int_L \frac{s^z}{\prod_{j=1}^{r}(\gamma_j - 1 - z)}\, dz \tag{6.16}$$

and L is an appropriate chosen contour of integration. See [Mathai 1993, Chapter 3] for the definition of G-function and its numerous properties and applications.

The joint $P_F \otimes P_F$ density of $X(r, n, \widetilde{m}, k)$ and $X(s, n, \widetilde{m}, k)$, $1 \le r < s \le n$, is given by

$$f_{r,s}(x,y) = c_{s-1} G_{s-r}\left(\frac{\overline{F}(y)}{\overline{F}(x)} \middle| \gamma_{r+1},\ldots,\gamma_s\right) \frac{G_r(\overline{F}(x) \mid \gamma_1,\ldots,\gamma_r)}{\overline{F}(x)},$$

$$\alpha < x < y < \beta \tag{6.17}$$

and the joint $\overset{3}{\underset{j=1}{\otimes}} P_F$ density function of $X(r, n, \widetilde{m}, k)$, $X(j, n, \widetilde{m}, k)$ and $X(s, n, \widetilde{m}, k)$, $1 \le r < j < s \le n$, may similarly be given as

$$f_{r,j,s}(x,t,y) = c_{s-1} \frac{1}{\overline{F}(x)} \frac{1}{\overline{F}(t)} G_{s-j}\left(\frac{\overline{F}(y)}{\overline{F}(t)} \middle| \gamma_{j+1},\ldots,\gamma_s\right)$$

$$\times G_{j-r}\left(\frac{\overline{F}(t)}{\overline{F}(x)} \middle| \gamma_{r+1},\ldots,\gamma_j\right) G_r(\overline{F}(x) \mid \gamma_1,\ldots,\gamma_r),$$

$$\alpha < x < t < y < \beta. \tag{6.18}$$

Hence the conditional P_F density function of $X(s,n,\widetilde{m},k)$ given $X(r,n,\widetilde{m},k) = x$, $1 \le r < s \le n$, is

$$f_{s|r}(y\,|\,x)=\frac{c_{s-1}}{c_{r-1}}G_{s-r}\left(\frac{\overline{F}(y)}{\overline{F}(x)}\Big|\gamma_{r+1},\ldots,\gamma_s\right)\frac{1}{\overline{F}(x)}I_{(x,\beta)}(y),$$

$$\alpha<x<y<\beta \qquad (6.19)$$

and the conditional P_F density function of $X(r,n,\widetilde{m},k)$ given $X(s,n,\widetilde{m},k)=y$, $1\le r<s\le n$, is

$$f_{r|s}(x\,|\,y)=\frac{G_{s-r}\left(\frac{\overline{F}(y)}{\overline{F}(x)}\Big|\gamma_{r+1},\ldots,\gamma_s\right)G_r(\overline{F}(x)\,|\,\gamma_1,\ldots,\gamma_r)}{\overline{F}(x)G_s(\overline{F}(y)\,|\,\gamma_1,\ldots,\gamma_s)}I_{(\alpha,y)}(x).$$

$$(6.20)$$

Also the conditional P_F density function of $X(j,n,\widetilde{m},k)$ given $X(r,n,\widetilde{m},k)=x$ and $X(s,n,\widetilde{m},k)=y$, $1\le r<j<s\le n$, is given by

$$f_{j|r,s}(t|x,y)=\frac{1}{\overline{F}(t)}$$

$$\times\frac{G_{s-j}\left(\frac{\overline{F}(y)}{\overline{F}(t)}\Big|\gamma_{j+1},\ldots,\gamma_s\right)G_{j-r}\left(\frac{\overline{F}(t)}{\overline{F}(x)}\Big|\gamma_{r+1},\ldots,\gamma_j\right)}{G_{s-r}\left(\frac{\overline{F}(y)}{\overline{F}(x)}\Big|\gamma_{r+1},\ldots,\gamma_s\right)}I_{(x,y)}(t).$$

$$(6.21)$$

7. Dual generalized order statistics [Burkschat *et al.*, 2003]:

The *pdf* of the dual generalized order statistics ($dgos$) $X'(r,n,\widetilde{m},k)$ is obtained by replacing $1-F(x)$ with $F(x)$.

Case I: $m_1=m_2=\ldots=m_{n-1}=m\,[\,m\text{-}dgos\,]$

The joint *pdf* of m-$dgos$ is [Burkschat *et al.*, 2003]:

$$f_{X'(1,n,m,k),\ldots,X'(n,n,m,k)}(x_1,\ldots,x_n)$$

$$=k\left(\prod_{r=1}^{n-1}\gamma_r\right)\left(\prod_{j=1}^{n-1}[F(x_j)]^m f(x_j)\right)[F(x_n)]^{k-1}f(x_n) \qquad (7.1)$$

for $F^{-1}(0) > x_1 \geq x_2 \geq \cdots \geq x_n > F^{-1}(1)$

Therefore, *pdf* of the r^{th} m-*dgos* is given by

$$f_{X'(r,n,m,k)}(x) = \frac{c_{r-1}}{(r-1)!}[F(x)]^{\gamma_r - 1} g_m^{r-1}[F(x)]f(x), \tag{7.2}$$

where

$$h_m(x) = \begin{cases} -\dfrac{1}{m+1}x^{m+1}, & m \neq -1 \\ -\log x, & m = -1 \end{cases} \tag{7.3}$$

$$g_m(x) = h_m(x) - h_m(1), \qquad x \in [0,1). \tag{7.4}$$

The joint *pdf* of $X'(r,n,m,k)$ and $X'(s,n,m,k)$, the r^{th} and s^{th} m-*dgos* is

$$f_{X'(r,n,m,k),X'(s,n,m,k)}(x,y)$$

$$= \frac{c_{s-1}}{(r-1)!(s-r-1)!}[F(x)]^m f(x) g_m^{r-1}[F(x)]$$

$$\times [h_m(F(y)) - h_m(F(x))]^{s-r-1}[F(y)]^{\gamma_s - 1} f(y),\ x > y. \tag{7.5}$$

Therefore, the conditional *pdf* of $X'(s,n,m,k)$ given $X'(r,n,m,k) = x$, $1 \leq r < s \leq n$, is

$$f_{X'(s,n,m,k)|X'(r,n,m,k)}(y\,|\,x)$$

$$= \frac{c_{s-1}}{(s-r-1)!c_{r-1}} \frac{[h_m(F(y)) - h_m(F(x))]^{s-r-1}}{[F(x)]^{\gamma_{r+1}}}$$

$$\times [F(y)]^{\gamma_s - 1} f(y), \quad x > y. \tag{7.6}$$

Case II: $\gamma_i \neq \gamma_j$, $i \neq j$

$$f_{X'(r,n,\tilde{m},k)}(x) = c_{r-1} f(x) \sum_{i=1}^{r} a_i(r)[F(x)]^{\gamma_i - 1} \tag{7.7}$$

The joint *pdf* of $X'(r,n,\tilde{m},k)$ and $X'(s,n,\tilde{m},k)$ for $1 \leq r < s \leq n$, is

$$f_{X'(r,n,\tilde{m},k),X'(s,n,\tilde{m},k)}(x,y)$$

$$= c_{s-1} \sum_{i=r+1}^{s} a_i^{(r)}(s) \left[\frac{F(y)}{F(x)}\right]^{\gamma_i} \sum_{i=1}^{r} a_i(r)[F(x)]^{\gamma_i} \frac{f(x)}{F(x)} \frac{f(y)}{F(y)}. \quad (7.8)$$

The joint *pdf* of $X'(r,n,\tilde{m},k)$, $X'(j,n,\tilde{m},k)$ and $X'(s,n,\tilde{m},k)$, $1 \leq r < j < s \leq n$, may similarly be given as

$$f_{X'(r,n,\tilde{m},k),X'(j,n,\tilde{m},k),X'(s,n,\tilde{m},k)}(x,t,y)$$

$$= c_{s-1} \left(\sum_{i=1}^{r} a_i(r)(F(x))^{\gamma_i} \right) \left(\sum_{i=r+1}^{j} a_i^{(r)}(j) \left[\frac{F(t)}{F(x)}\right]^{\gamma_i} \right)$$

$$\times \left(\sum_{i=j+1}^{s} a_i^{(j)}(s) \left[\frac{F(y)}{F(t)}\right]^{\gamma_i} \right) \frac{f(x)}{F(x)} \frac{f(t)}{F(t)} \frac{f(y)}{F(y)},$$

$$\alpha < y < t < x < \beta. \quad (7.9)$$

Therefore, the conditional *pdf* of $X'(s,n,\tilde{m},k)$ given $X'(r,n,\tilde{m},k) = x$, $1 \leq r < s \leq n$, is

$$f_{X'(s,n,\tilde{m},k)|X'(r,n,\tilde{m},k)}(y \mid x)$$

$$= \frac{c_{s-1}}{c_{r-1}} \sum_{i=r+1}^{s} a_i^{(r)}(s) \frac{[F(y)]^{\gamma_i - 1}}{[F(x)]^{\gamma_i}} f(y), \quad x > y. \quad (7.10)$$

8. Some Special functions

8.1. Meijer's G-function

Meijer's G-functions are defined by the Mellin-Barnes type integral

$$G_{p,q}^{m,n}\left[x \,\middle|\, \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_p \end{matrix}\right]$$

$$= \frac{1}{2\pi i} \oint_L \frac{\prod_{j=1}^{m} \Gamma(b_j - z) \prod_{j=1}^{n} \Gamma(1 - a_j + z)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j + z) \prod_{j=n+1}^{p} \Gamma(a_j - z)} x^z \, dz, \quad (8.1)$$

where $0 \le n \le p$, $0 \le m \le q$, $|x|<1$ and $a_1,\ldots,a_p, b_1,\ldots,b_p \in C$. L is an appropriately chosen integration path in the complex plane. For detailed accounts on Meijer's G-functions, we refer to Mathai (1993).

In the situation of generalized order statistics, the representation of the respective Meijer's G-function simplifies considerably because the parameters a_j, b_j are given by $a_j = \gamma_j$ and $b_j = \gamma_j - 1$. Hence, the integrand of the Meijer's G-function is given in (8.1) becomes

$$G_{r,r}^{r,0}\left(x \left| \begin{matrix} \gamma_1,\ldots,\gamma_r \\ \gamma_1 - 1,\ldots,\gamma_r - 1 \end{matrix} \right. \right) = \frac{1}{2\pi i} \oint_L \frac{x^z}{\prod_{j=1}^{r} (\gamma_j - 1 - z)} dz, \quad |x|<1. \quad (8.2)$$

Since in our context, Meijer's G-function depends only on $\gamma_1,\ldots,\gamma_n$, we use , for brevity, the notation

$$G_{r,r}^{r,0}[x \mid \gamma_1,\ldots,\gamma_r] = G_{r,r}^{r,0}\left[x \left| \begin{matrix} \gamma_1,\ldots,\gamma_r \\ \gamma_1 - 1,\ldots,\gamma_r - 1 \end{matrix} \right. \right], \quad x \in [0,1].$$

Meijer's G-functions satisfy many important recurrence relations which is given below

(i) $G_1(x \mid \gamma_1) = x^{\gamma_1 - 1}$

(ii) $(\gamma_r - \gamma_1) G_r(x \mid \gamma_1,\ldots,\gamma_r) = [G_{r-1}(x \mid \gamma_1,\ldots,\gamma_{r-1})$

$$- G_{r-1}(x \mid \gamma_2,\ldots,\gamma_r)]$$

(iii) $x^a G_r(x \mid \gamma_1,\ldots,\gamma_r) = G_r(x \mid \gamma_1 + a,\ldots,\gamma_r + a), \quad a \in R$

(iv) $\lim_{x \to 1-} G_r(x \mid \gamma_1,\ldots,\gamma_r) = \begin{cases} 1, & r = 1 \\ 0, & r \ge 2 \end{cases}$

and

$$\lim_{x\to 0+} G_r(x\mid\gamma_1,\ldots,\gamma_r)=\begin{cases}0, & if\ \gamma_{1:r}>1\\ \prod\limits_{\substack{j=1\\ j\neq l}}\frac{1}{\gamma_j-\gamma_l}, & if\gamma_{1:r}=1<\gamma_{2:r}\\ \infty, & if\ \gamma_{1:r}=\gamma_{2:r}=1\ \ or\ \gamma_{1:r}<1\end{cases}$$

where

$\gamma_{1:r}=\min[\gamma_1,\ldots,\gamma_r]$ and $l=\max[1\le j\le r:\gamma_j=\gamma_{1:r}]$

(v) $$\frac{d}{dx}G_r(x\mid\gamma_1,\ldots,\gamma_r)=\frac{1}{x}\big[(\gamma_r-1)G_r(x\mid\gamma_1,\ldots,\gamma_r)\\ -G_{r-1}(x\mid\gamma_1,\ldots,\gamma_{r-1})\big]$$

(vi) $$\frac{d}{dx}G_r(x\mid\gamma_1,\ldots,\gamma_r)=\frac{1}{x}\big[(\gamma_1-1)G_r(x\mid\gamma_1,\ldots,\gamma_r)\\ -G_{r-1}(x\mid\gamma_2,\ldots,\gamma_r)\big]$$

Proof: For the property (i), see Mathai (1993, p. 130), for property (ii), see Cramer and Kamps (2003), and for the property (iii), see Mathai (1993, p. 69). Property (iv) can easily be deduced from Lemma 2.2 of Cramer *et al.* (2004), whereas (v) and (vi) can be established from (8.2).

8.2. Formula for definite integral [Prudnikov *et al.,* 1986]:

(i) $$\int_0^a x^{\alpha-1}(a^\delta-x^\delta)^{\beta-1}[\log x]^n dx=\frac{a^{\delta(\beta-1)}}{\delta}\frac{\partial^n}{\partial\alpha^n}\left[a^\alpha\,\mathrm{B}(\beta,\frac{\alpha}{\delta})\right],\quad a,\delta,\ \alpha,\ \beta>0$$

(ii) $$\int_0^1 x^{\alpha-1}[\log x]^n dx=\frac{(-1)^n n!}{\alpha^{n+1}}.$$

9. Some continuous distributions

I. Exponential distribution

A random variable X is said to have an exponential distribution if its *pdf* is given by

$$f(x) = \theta e^{-\theta x},\ 0 \le x < \infty;\ \theta > 0$$

with the *df*

$$F(x) = 1 - e^{-\theta x},\ 0 \le x < \infty;\ \theta > 0.$$

The exponential distribution plays an important role in describing a large class of phenomena particularly in the area of reliability theory. The exponential distribution has many other applications. In fact, whenever a continuous random variable X assuming non-negative values satisfies the assumption,

$$P(X > s + t \mid X > s) = P(X > t) \text{ for all } s \text{ and } t,$$

then X will have an exponential distribution. This is particularly a very appropriate failure law when present does not depend on the past, for example, in studying the life of a bulb etc.

II. Weibull Distribution

A random variable X is said to have a Weibull distribution if its *pdf* is given by

$$f(x) = \theta p x^{p-1} e^{-\theta x^p},\ 0 \le x < \infty;\ \theta > 0,\ p > 0$$

and the *df* is given by

$$F(x) = 1 - e^{-\theta x^p},\ 0 \le x < \infty;\ \theta > 0,\ p > 0$$

Weibull distribution is widely used in reliability and quality control. The distribution is also useful in cases where the conditions of strict randomness of exponential distribution are not satisfied. It is sometimes used as a tolerance distribution in the analysis of quantal response data.

If we put $p = 1$ in Weibull distribution, we get the *pdf* of an **exponential distribution**.

If we put $p = 2$, it gives the *pdf* of **Rayleigh distribution.**

If X has a Weibull distribution, then the *pdf* of $Y = -\log(\theta X^p)$ is

$$f(y) = e^{-y} e^{-e^{-y}}$$

which is a form of Gumbel (Extreme Value Type-I) distribution.

III. Pareto distribution

A random variable X is said to have the Pareto distribution if its *pdf* $f(x)$ and the *df* $F(x)$ are of the form given below:

$$f(x) = p a^p x^{-(p+1)}, \ a \leq x < \infty; \ a, p > 0$$

$$F(x) = 1 - a^p x^{-p}, \ a \leq x < \infty; a, p > 0.$$

Many socio-economic and naturally occurring quantities are distributed according to the Pareto law. For example, distribution of city population sizes, personal income etc.

IV. Power function distribution

A random variable X is said to have a power function distribution if its *pdf* and the *df* are of the form given below:

$$f(x) = p a^{-p} x^{p-1}, \ 0 \leq x < a; \ a, p > 0$$

$$F(x) = a^{-p} x^p, \ 0 \leq x < a; \ a, p > 0$$

The power function distribution is used to approximate representation of the lower tail of the distribution of random variable having fixed lower bound. It may be noted that if X has a power function distribution, then $Y = \dfrac{1}{X}$ has a Pareto distribution.

V. Beta distribution

a) Beta distribution of the first kind

A random variable X is said to have the beta distribution of the first kind if its *pdf* is of the form

$$f(x)=\frac{1}{B(p,q)}x^{p-1}(1-x)^{q-1},\ \ 0\le x\le 1;\ \ p,q>0.$$

Beta distribution arises as the distribution of an ordered variable from a uniform distribution. Suppose $X_{r:n}$ is an ordered sample from $U(0, 1)$, then $X_{r:n}$ is distributed as beta $B(r, n-r+1)$. The standard uniform distribution $U(0,1)$ is the special case of beta distribution of first kind obtained by putting the exponents p and q equal to 1. If $q=1$, the distribution reduces to the power function distribution.

b) Beta distribution of the second kind

The continuous random variable X which is distributed according to probability law:

$$f(x)=\frac{1}{B(p,q)}\frac{x^{p-1}}{(1+x)^{p+q}},\ 0<x<\infty\,;p,q>0$$

is known as a beta variate of the second kind with parameters p and q.

The beta distribution is one of the most frequently employed distributions to fit theoretical distributions. Beta distribution may be applied directly to the analysis of Markov processes with "uncertain" transition probabilities.

VI. Logistic distribution

A random variable X is said to have logistic distribution if its *pdf* is given by

$$f(x) = \frac{e^{-x}}{(1+e^{-x})^2}, \ -\infty < x < \infty$$

and the *df* is given by

$$F(x) = 1 - (1+e^{x})^{-1}, \ -\infty < x < \infty.$$

A plausible characterizing property of this distribution is that $F(x)[1-F(x)] = f(x)$.

VII. Extreme value distribution

A random variable X is said to have an extreme value distribution of type I if its *pdf* is given by

$$f(x) = e^{x}\exp[-e^{x}], \ -\infty < x < \infty$$

and the *df* is given by

$$F(x) = 1 - \exp[-e^{x}], \ -\infty < x < \infty$$

VIII. Inverse Weibull distribution

A random variable X is said to have an extreme value distribution of type II (Inverse Weibull distribution) if its *pdf* is given by

$$f(x) = \theta p x^{-(p+1)} e^{-\theta x^{-p}} \ , 0 < x < \infty; \ \theta, \ p > 0$$

and the *df* is given by

$$F(x) = e^{-\theta x^{-p}} \ , 0 < x < \infty; \ \theta, \ p > 0$$

The extreme value distribution is applied very much in natural phenomenon such as rain fall, floods, wind gusts and air pollution. If we put $p = 1$ in Inverse Weibull distribution, we get the *pdf* of **inverted exponential distribution**.

IX. Generalized (reflected) exponential distribution

A random variable X is said to have a generalized (reflected) exponential distribution to be denoted as $X \sim$ *gen(ref)exp* (α) if its *pdf* is given by

$$f(x) = \alpha e^{\alpha x}[1-(m+1)e^{\alpha x}]^{\frac{1}{m+1}-1},$$

$$-\infty < x < \beta;\ \alpha > 0, m > -1$$

with the *df*

$$\overline{F}(x) = [1-(m+1)e^{\alpha x}]^{\frac{1}{m+1}}, \quad -\infty < x < \beta;\ \alpha > 0, m > -1$$

where $\beta = -\frac{1}{\alpha}\ln(m+1)$.

X. Generalized power distribution

A random variable X is said to have a generalized power distribution to be denoted as $X \sim$ *genpow* (α) if its *pdf* is given by

$$f(x) = \alpha x^{\alpha-1}[1-(m+1)x^{\alpha}]^{\frac{1}{m+1}-1}, \quad 0 \le x < \beta;\ \alpha > 0, m > -1$$

with the *df*

$$\overline{F}(x) = [1-(m+1)x^{\alpha}]^{\frac{1}{m+1}}, \quad 0 \le x < \beta;\ \alpha > 0, m > -1,$$

where $\beta = (m+1)^{-\frac{1}{\alpha}}$.

XI. Generalized exponential distribution

A random variable X is said to have a generalized exponential distribution to be denoted as $X \sim$ *genexp* (α) if its *pdf* is given by

$$f(x)=\alpha e^{-\alpha x}[1-(m+1)e^{-\alpha x}]^{\frac{1}{m+1}-1}, \quad \beta \le x < \infty;\ \alpha > 0,\ m > -1$$

with the *df*

$$F(x)=[1-(m+1)e^{-\alpha x}]^{\frac{1}{m+1}}, \quad \beta \le x < \infty;\ \alpha > 0, m > -1,$$

where $\beta = \frac{1}{\alpha}\ln(m+1)$.

XII. Generalized Pareto distribution

A random variable X is said to have a generalized Pareto distribution to be denoted as $X \sim genPar\ (\alpha)$ if its *pdf* is given by

$$f(x)=\alpha x^{-\alpha-1}[1-(m+1)x^{-\alpha}]^{\frac{1}{m+1}-1}, \quad \beta \le x < \infty;\ \alpha > 0, m > -1$$

with the *df*

$$F(x)=[1-(m+1)x^{-\alpha}]^{\frac{1}{m+1}}, \quad \beta \le x < \infty;\ \alpha > 0, m > -1,$$

where $\beta = (m+1)^{\frac{1}{\alpha}}$.

XIII. Burr distribution

Let X be a continuous random variable, then different forms of cumulative distribution function of X are listed below (Johnson *et al.*, 1994):

Type-I : $F(x)=x,\ 0<x<1$

Type-II : $F(x)=(1+e^{-x})^{-k}, \quad -\infty < x < \infty$

Type-III : $F(x)=(1+x^{-c})^{-k}, \quad 0 \le x < \infty$

Type-IV : $F(x)=\left[1+\left(\frac{c-x}{x}\right)^{1/c}\right]^{-k}, \quad 0 \le x \le c$

Type-V : $F(x) = [1 + ce^{-\tan x}]^{-k}, \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$

Type-VI : $F(x) = [1 + ce^{-k\sinh x}]^{-k}, \quad -\infty < x < \infty$

Type-VII : $F(x) = 2^{-k}(1 + \tanh x)^k, \quad -\infty < x < \infty$

Type-VIII : $F(x) = \left(\frac{2}{\pi}\tan^{-1} e^x\right), \quad -\infty < x < \infty$

Type-IX : $F(x) = 1 - \frac{2}{c[(1+e^x)^k - 1] + 2}, \quad -\infty < x < \infty$

Type-X : $F(x) = (1 + e^{-x^2})^k, \quad 0 \le x < \infty$

Type-XI : $F(x) = \left(x - \frac{1}{2\pi}\sin 2\pi x\right)^k, \quad 0 \le x \le 1$

Type-XII : $F(x) = 1 - (1 + x^c)^{-k}, \quad 0 \le x < \infty,$

where k and c are positive parameters.

Special attention is given to Burr type XII, whose *pdf* is given as:

$$f(x) = kcx^{c-1}(1 + x^c)^{-(k+1)}, \; 0 \le x < \infty; \; k, c > 0.$$

This distribution is frequently used for the purpose of graduation and in reliability theory. At $c = 1$, it is called **Lomax distribution** whereas at $k = 1$, it is known as **Log-logistic distribution.**

XIV. Cauchy distribution

The special form of the Pearson type VII distribution, with the *pdf*

$$f(x) = \frac{1}{\pi}\frac{1}{\lambda[1 + \{(x-\theta)/\lambda\}^2]}, -\infty < x < \infty; \; \lambda > 0, -\infty < \theta < \infty$$

is called the Cauchy distribution.

The *df* is given by

$$F(x)=\frac{1}{2}+\frac{1}{\pi}\tan^{-1}\left(\frac{x-\theta}{\lambda}\right),\ -\infty<x<\infty;\ \lambda>0,-\infty<\theta<\infty.$$

The distribution is symmetrical about $x=\theta$. The distribution does not possess finite moments of order greater than or equal to 1, and so does not possess a finite expected value or standard deviation. However, θ and λ are location and scale parameters, respectively and may be regarded as being analogous to mean and standard deviation.

There is no standard form of the Cauchy distribution, as it is not possible to standardize without using (finite) values of mean and standard deviation, which does not exist in this case. However, a standard form is obtained by putting $\theta=0,\lambda=1$. The standard probability density function is given by

$$f(x)=\frac{1}{\pi}\frac{1}{(1+x^2)}\ ,\quad -\infty<x<\infty$$

and the standard cumulative distribution function is

$$F(x)=\frac{1}{2}+\frac{1}{\pi}\tan^{-1}x,\quad -\infty<x<\infty.$$

It may be noted that if X and Y are standard normal variate then their ratio is Cauchy variate.

10. Abstract of thesis

Characterization results are those which shed light on modeling sequences of certain distributional assumptions and those which have potential for development of hypothesis testing for model assumptions.

Continuous distributions have been characterized through conditional expectations of order statistics, record values, generalized order statistics and dual generalized order statistics.

Main contributors for characterizations through order statistics are:

Ferguson (1967), Shanbhag (1970), Hamdan (1972), Beg and Kirmani (1978), Khan and Khan (1986), Khan and Beg (1987), Khan and Ali (1987), Nagaraja (1988), Khan and Abu-Salih (1989), Ouyang and Wu (1996), Franco and Ruiz (1995, 1997), Blaquez and Rebollo (1997), Wesolowski and Ahsanullah (1997), Dembińska and Wesolowski (1998) and Khan and Abouammoh (2000), Lee *et al.* (2002), Balakrishnan and Akhundov (2003) and Khan *et al.* (2007) amongst others.

Through record values, contribution of Franco and Ruiz (1996), Nagaraja and Nevzorov (1997), Ahsanullah and Wesolowski (1998), Dembińska and Wesolowski (2000), Raqab (2002), Athar *et al.*(2003), Lee (2003) and Khan and Khan (2009) are significant.

Concept of generalized order statistics (*gos*) was given by Kamps (1995). Since many ordered variables like order statistics, record values and k – record values are special cases of generalized order statistics, therefore characterization through generalized order statistics is of special interest. Main contributors are Keseling (1999), Bieniek and Szynal (2003), Cramer *et al.* (2004), Khan and Alzaid (2004), Raqab and Abu-Lawi (2004), Ahsanullah and Raqab (2004), Ahsanullah (2005), Beg and Ahsanullah (2006, 2008), Khan *et al.* (2006), Bieniek (2007, 2009), Samuel (2008), Ahsanullah *et al.* (2009) and Khan and Khan (2011) amongst others.

Also, distributions have been characterized through conditional spacings of order statistics , record statistics and generalized order statistics by Navarro *et al.*(1998), Cramer and Kamps (2001), Kamps and Keseling (2003), Cramer *et al.* (2003) and Khan *et al.* (2009a, 2010b, 2011) amongst others.

Using the concept of *gos*, Burkschat *et al.* (2003) introduced the concept of the dual generalized order statistics (*dgos*) that enables a common approach to descendingly ordered random variables like reversed ordered order statistics, lower record values etc. The main contributors are Ahsanullah (2004), Mbah and Ahsanullah (2007), Malinowska and Szynal (2008), Khan *et al.* (2009b, 2010a,c, 2012c), Shawky and Bakoban (2009) and Tavanagar (2011) amongst others.

The research work presented in this thesis is based on characterization of distributions through conditional expectation of ordered statistics. This thesis is spread over in five chapters. A comprehensive bibliography has also been given at the end, which has been referred during the research work.

Chapter I is expository in nature and provides a brief review of the concepts and results concerning order statistics, upper records, lower records, generalized order statistics and dual generalized order statistics. Some continuous distributions, which have been characterized are discussed as well.

In **Chapter II, a** family of continuous distribution has been characterized through contrast of conditional expectations of order statistics, record statistics and generalized order statistics conditioned respectively on non-adjacent order statistic, record statistic and generalized order statistic, extending the earlier known results. In particular, it has been shown that for $1 \le m < r < s \le n$,

(i) $$\sum_{i=r}^{s} b_i E[h(X_{l:n}) \mid X_{m:n} = x] = \frac{1}{a}\sum_{i=r}^{s} b_i \sum_{j=m}^{l-1} \frac{1}{(n-j)}, \quad l = i-1, i$$

if and only if

$$F(x) = 1 - e^{-ah(x)}, \quad x \in (\alpha, \beta); \ a > 0,$$

where b_i are real numbers $r \le i \le s$, satisfying $\sum_{i=r}^{s} b_i = 0$, $b_i \ne 0$ for some i and $h(x)$ is a monotonic , increasing and differentiable function of x. $X_{r:n}$ is the r^{th} order statistic in a sample of size n.

(ii) For $1 \le m < r < s$,

$$\sum_{i=r}^{s} b_i E[h(X_{U(l)}) \mid X_{U(m)} = x] = \frac{1}{a} \sum_{i=r}^{s} l b_i \,, l = i-1, i$$

if and only if

$$F(x) = 1 - e^{-ah(x)},\ x \in (\alpha, \beta); a > 0$$

$X_{U(m)}$ is the m^{th} upper record value from a continuous population.

(iii) For $1 \le r < s < t \le n$,

$$\sum_{i=s}^{t} b_i E[h\{X(l,n,m,k)\} \mid X(r,n,m,k) = x] = \frac{1}{a} \sum_{i=s}^{t} b_l \sum_{j=r+1}^{i} \frac{1}{\gamma_j}, l = i-1, i$$

if and only if

$$F(x) = 1 - e^{-ah(x)},\ x \in (\alpha, \beta);\ a > 0$$

where $X(r,n,m,k)$ is the $r^{th} - gos$ from a continuous population.

Some of important deductions for gos, order statistics and records are also discussed.

Chapter III embodies the results based on characterization of a family of continuous distributions by considering conditional expectations of generalized order statistics. Further, using Meijer's G-function, conditional expectation of generalized order statistics based on two non-adjacent generalized order statistics has been used to characterize a general form of distribution. Precisely, it has been shown that for $1 \le r < s \le n$,

(i) $E[X^{\alpha}(s,n,m,k) \mid X(l,n,m,k)=x]=a^{*}_{s|l}x^{\alpha}+b^{*}_{s|l}, l=r,r+1$

if and only if

$$\overline{F}(x)=[1-(m+1)x^{\alpha}]^{\frac{1}{m+1}}, 0 \le x \le \beta;\ \alpha>0$$

where

$$\beta=(m+1)^{-\frac{1}{\alpha}}, a^{*}_{s|r}=\frac{\gamma_s}{\gamma_r} \text{ and } b^{*}_{s|r}=\frac{1}{m+1}[1-a^{*}_{s|r}]$$

(ii) For $1 \le r < s < t \le n$,

$$E[X^{\alpha}(t,n,m,k) \mid X(r,n,m,k)=x]$$

$$=a^{*}_{t|s}E[X^{\alpha}(s,n,m,k) \mid X(r,n,m,k)=x]+b^{*}_{t|s}$$

if and only if

$$\overline{F}(x)=[1-(m+1)x^{\alpha}]^{\frac{1}{m+1}}, 0 \le x \le \beta; \alpha>0$$

where

$$\beta=(m+1)^{-\frac{1}{\alpha}},\ a^{*}_{t|s}=\frac{\gamma_t}{\gamma_s} \text{ and } b^{*}_{t|s}=\frac{1}{m+1}[1-a^{*}_{t|s}]$$

(iii) $E[e^{\alpha X(s,n,m,k)} \mid X(l,n,m,k)=x]=a^{*}_{s|l}e^{\alpha x}+b^{*}_{s|l}, l=r,r+1$

if and only if

$$\overline{F}(x)=[1-(m+1)e^{\alpha x}]^{\frac{1}{m+1}}, -\infty < x \le \beta; \alpha>0$$

where

$$\beta = -\frac{1}{\alpha}\ln(m+1) \quad , \; a^{*}_{s|r} = \frac{\gamma_s}{\gamma_r} \quad \text{and} \quad b^{*}_{s|r} = \frac{1}{m+1}[1 - a^{*}_{s|r}]$$

(iv) $E[e^{\alpha X(t,n,m,k)} \mid X(r,n,m,k) = x]$

$$= a^{*}_{t|s} E[e^{\alpha X(s,n,m,k)} \mid X(r,n,m,k) = x] + b^{*}_{t|s}$$

if and only if

$$\bar{F}(x) = [1 - (m+1)e^{\alpha x}]^{\frac{1}{m+1}}, -\infty < x \le \beta;\ \alpha > 0 \quad ,$$

where

$$\beta = -\frac{1}{\alpha}\ln(m+1),\ a^{*}_{t|s} = \frac{\gamma_t}{\gamma_s} \text{ and } b^{*}_{t|s} = \frac{1}{m+1}[1 - a^{*}_{t|s}].$$

Chapter IV is based on the characterization of generalized families of continuous distributions through conditional expectation of lower record values and dual generalized order statistics (*dgos*) conditioned on a non-adjacent *dgos*, extending the result of Khan *et al.* (2010a). For $1 \le r < s < t$,

(i) $E[h\{X_{L(t)}\} - h\{X_{L(s)}\} \mid X_{L(r)} = x] = \frac{(t-s)}{a}$

if and only if

$$F(x) = e^{-ah(x)},\ x \in (\alpha, \beta); a > 0$$

(ii) For $1 \le r < s < t \le n$,

$$E[h\{X'(t,n,m,k)\} - h\{X'(s,n,m,k)\} \mid X'(r,n,m,k) = x]$$

$$= \frac{1}{a} \sum_{j=s+1}^{t} \frac{1}{\gamma_j}$$

if and only if

$$F(x) = e^{-a h(x)},\ \ x \in (\alpha, \beta);\ \ a > 0,$$

where $h(x)$ is a monotonic, non- increasing and differentiable function of x such that $h(x) \to 0$ as $x \to \beta$ and $h(x)F(x) \to 0$ as $x \to \alpha$.

(iii) For $1 \le r < s < t \le n$,

$$E[h\{X'(t,n,m,k)\} \mid X'(r,n,m,k) = x]$$

$$= a^*_{t|s} E[h\{X'(s,n,m,k)\} \mid X'(r,n,m,k) = x] + b^*_{t|s}$$

if and only if

$$F(x) = [ah(x) + b]^c, \quad x \in (\alpha, \beta),$$

where

$$a^*_{t|s} = \prod_{j=s+1}^{t} \frac{c\gamma_j}{1 + c\gamma_j} \text{ and } b^*_{t|s} = -\frac{b}{a}(1 - a^*_{t|s}).$$

Chapter V deals with exact expressions for single and product moments of record statistics for two parameters Burr type XII distribution. The means, variances and covariances of the record statistics are computed for various values of the shape parameters. These values are then used to compute the coefficients of the best linear unbiased estimators of the location and scale parameters. The variances of these estimators are also obtained. The predictors of the future record statistics are discussed.

CHAPTER-II

CHARACTERIZATION OF PROBABILITY DISTRIBUTIONS THROUGH CONTRAST OF ORDER STATISTICS, RECORD AND GENERALIZED ORDER STATISTICS

1. Introduction

Distributions have been characterized through the conditional expectations of order statistics, record statistics and generalized order statistics by Ferguson (1967) , Khan and Abu-Salih (1989), Franco and Ruiz (1997), Nagaraja (1997), Khan and Abouammoh (2000), Wu and Lee (2001), Lee (2003), Athar *et al.*(2003), Khan and Alzaid (2004) and Khan *et al.* (2006) amongst others.

Also, distributions have been characterized through conditional spacings of order statistics , record statistics and generalized order statistics by Navarro *et al.*(1998), Cramer and Kamps (2001), Kamps and Keseling (2003), Cramer *et al.* (2003) and Khan *et al.* (2009a, 2010b, 2011).

We, in this chapter have considered contrast of conditional expectations in characterizing the distributions, thus generalizing the earlier known results on order statistics, record statistics and generalized order statistics on conditional expectations and spacings.

2. Characterization theorem based on order statistics

Let $X_{r:n}$ be the r^{th} order statistic from a sample of size n from a continuous population having the probability density function (pdf) $f(x)$ and distribution function (df) $F(x)$. Then the conditional pdf of $X_{s:n}$ given $X_{r:n}=x,\ 1\le r<s\le n$, is [David and Nagaraja, 2003]:

Part of the results of this chapter are contained in Faizan and Khan (2013) and Khan *et al.* (2012a, 2013b).

$$\left[\frac{(n-r)!}{(s-r-1)!(n-s)!}\right]\frac{[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s}}{[1-F(x)]^{n-r}}f(y),$$

$$x \le y. \qquad (2.1)$$

Theorem 2.1: Let X be an absolutely continuous random variable with the *df* $F(x)$ and the *pdf* $f(x)$ over the support (α,β), where α and β may be finite or infinite. Then for $1 \le m < r < s \le n$,

$$\sum_{i=r}^{s} b_i E[h(X_{l:n}) \mid X_{m:n}=x] = \frac{1}{a}\sum_{i=r}^{s} b_i \sum_{j=m}^{l-1}\frac{1}{(n-j)}, \quad l=i-1, i \qquad (2.2)$$

if and only if

$$F(x)=1-e^{-ah(x)}, \quad x\in(\alpha,\beta); \; a>0, \qquad (2.3)$$

where b_i are real numbers $r \le i \le s$, satisfying $\sum_{i=r}^{s} b_i = 0$, $b_i \ne 0$ for some i and $h(x)$ is a monotonic, increasing and differentiable function of x such that (2.3) is a *df* .

Proof: First, we will prove (2.3) implies (2.2).We have (Khan and Abouammoh, 2000)

$$E[h(X_{i:n}) \mid X_{m:n}=x] = h(x)+\frac{1}{a}\sum_{j=m}^{i-1}\frac{1}{(n-j)}$$

for $F(x)=1-e^{-ah(x)}$, $x\in(\alpha,\beta)$; $a>0$.

Therefore,

$$\sum_{i=r}^{s} b_i E[h(X_{i:n}) \mid X_{m:n}=x] = \sum_{i=r}^{s} b_i\left[h(x)+\frac{1}{a}\sum_{j=m}^{i-1}\frac{1}{(n-j)}\right]$$

$$=\frac{1}{a}\sum_{i=r}^{s} b_i\sum_{j=m}^{i-1}\frac{1}{(n-j)}, \text{ as } \sum_{i=r}^{s} b_i = 0.$$

Hence the 'if' part.

To prove the sufficiency part, let $d = \frac{1}{a}\sum_{i=r}^{s} b_i \sum_{j=m}^{i-1} \frac{1}{(n-j)}$.

Then,

$$\sum_{i=r}^{s} b_i E[h(X_{i:n}) \mid X_{m:n} = x] = d \tag{2.4}$$

or,

$$\sum_{i=r}^{s} b_i \frac{(n-m)!}{(i-m-1)!(n-i)!} \int_x^{\beta} h(y)[F(y)-F(x)]^{i-m-1}$$

$$\times [1-F(y)]^{n-i} f(y)dy = d[1-F(x)]^{n-m}. \tag{2.5}$$

Integrating left hand side of (2.5) by parts treating $[1-F(y)]^{n-i} f(y)$ for integration and $h(y)[F(y)-F(x)]^{i-m-1}$ for differentiation, we get

$$\sum_{i=r}^{s} b_i \frac{(n-m)!}{(i-m-2)!(n-i+1)!} \int_x^{\beta} h(y)[F(y)-F(x)]^{i-m-2}$$

$$\times [1-F(y)]^{n-i+1} f(y)dy + \sum_{i=r}^{s} b_i \frac{(n-m)!}{(i-m-1)!(n-i+1)!}$$

$$\times \int_x^{\beta} h'(y)[F(y)-F(x)]^{i-m-1}[1-F(y)]^{n-i+1} dy$$

$$= d[1-F(x)]^{n-m}. \tag{2.6}$$

We can write equation (2.6) as

$$\sum_{i=r}^{s} b_i \frac{(n-m)!}{(i-m-1)!(n-i+1)!} \int_x^{\beta} h'(y)[F(y)-F(x)]^{i-m-1}$$

$$\times [1-F(y)]^{n-i+1} dy$$

$$= (d - d_1)[1 - F(x)]^{n-m} \text{ in view of (2.2)} \tag{2.7}$$

where

$$d_1 = \frac{1}{a}\sum_{i=r}^{s} b_i \sum_{j=m}^{i-2} \frac{1}{(n-j)}.$$

That is,

$$\sum_{i=r}^{s} b_i \frac{(n-m)!}{(i-m-1)!(n-i+1)!} \int_x^{\beta} h'(y)[F(y)-F(x)]^{i-m-1}$$

$$\times [1-F(y)]^{n-i+1} dy = \frac{1}{a}\sum_{i=r}^{s} b_i \frac{1}{(n-i+1)}[1-F(x)]^{n-m}. \tag{2.8}$$

Differentiating (2.8) $(i-m)$ times both the sides $w.r.t.\ x$, we get

$$\sum_{i=r}^{s} b_i \frac{[1-F(x)]^{n-i}}{(n-i+1)!}[1-F(x)]h'(x) = \sum_{i=r}^{s} b_i \frac{[1-F(x)]^{n-i}}{(n-i+1)!}\frac{f(x)}{a}$$

$$\left[[1-F(x)]h'(x) - \frac{f(x)}{a}\right]\sum_{i=r}^{s} b_i \frac{[1-F(x)]^{n-i}}{(n-i+1)!} = 0$$

$$[1-F(x)]h'(x) = \frac{f(x)}{a}, \text{ as } \sum_{i=r}^{s} b_i \frac{[1-F(x)]^{n-i}}{(n-i+1)!} \neq 0.$$

That is,

$$F(x) = 1 - \exp[-ah(x)]; \quad a > 0$$

and hence the Theorem.

Remark 2.1: This result has appeared in Khan *et al.* (2012a). Putting $b_s = 1$ and $b_r = -1$ in Theorem 2.1, we get characterizing results as obtained by Khan *et al.* (2009a). Also at $m = r$, it reduces to the result as obtained by Khan and Abouammoh (2000).

Table 2.1: Examples based on the distribution function $F(x)=1-e^{-ah(x)};\ a>0$.

Distribution	$F(x)$	a	$h(x)$
Exponential	$1-e^{-\theta x}$ $0<x<\infty$	θ	x
Weibull	$1-e^{-\theta x^{p}}$ $0<x<\infty$	θ	x^{p}
Pareto	$1-\left(\frac{x}{a}\right)^{-p}$ $a<x<\infty$	p	$\log\left(\frac{x}{a}\right)$
Lomax	$1-(1+x)^{-k}$ $0<x<\infty$	k	$\log(1+x)$
Gompertz	$1-\exp[-\frac{\lambda}{\mu}(e^{\mu x}-1)]$ $0<x<\infty$	$\frac{\lambda}{\mu}$	$e^{\mu x}-1$
Beta of the I kind	$1-(1-x)^{p}$ $0<x<1$	p	$-\log(1-x)$
Beta of the II kind	$1-(1+x)^{-1}$ $0<x<\infty$	1	$\log(1+x)$
Extreme value I	$1-\exp[-e^{x}]$ $-\infty<x<\infty$	1	e^{x}
Log logistic	$1-(1+x^{c})^{-1}$ $0<x<\infty$	1	$\log(1+x^{c})$
Burr Type IX	$\left[\frac{c\{(1+e^{x})^{k}-1\}}{2}+1\right]^{-1}$ $-\infty<x<\infty$	1	$\log\left[\frac{c\{(1+e^{x})^{k}-1\}}{2}+1\right]$
Burr Type XII	$1-(1+x^{c})^{-k}$ $0<x<\infty$	k	$\log(1+x^{c})$

3. Characterization theorem based on record statistics

Let $\{X_n, n \ge 1\}$ be a sequence of independent and identically distributed (iid) continuous random variables with the distribution function $df\ F(x)$ and the probability density function $pdf\ f(x)$. Let $X_{U(s)}$ be the $s-th$ upper record value, then the conditional pdf of $X_{U(s)}$ given $X_{U(r)} = x,\ 1 \le r < s$, is [Ahsanullah, 1995]:

$$f[X_{U(s)} \mid X_{U(r)} = x] = \frac{1}{\Gamma(s-r)}[-\ln \bar{F}(y) + \ln \bar{F}(x)]^{s-r-1} \frac{f(y)}{\bar{F}(x)}, \tag{3.1}$$

where $\bar{F}(x) = P(X > x) = 1 - F(x)$ and $\ln$ is log with base e.

Theorem 3.1: Let X be an absolutely continuous random variable with the $df\ F(x)$ and the $pdf\ f(x)$ over the support (α, β), where α and β may be finite or infinite. Then for $1 \le m < r < s$,

$$\sum_{i=r}^{s} b_i E[h(X_{U(l)}) \mid X_{U(m)} = x] = \frac{1}{a} \sum_{i=r}^{s} l b_i, \quad l = i-1, i \tag{3.2}$$

if and only if

$$F(x) = 1 - e^{-ah(x)}, \quad x \in (\alpha, \beta); \quad a > 0, \tag{3.3}$$

where b_i are real numbers $r \le i \le s$, satisfying $\sum_{i=r}^{s} b_i = 0$, $b_i \ne 0$ for some i and $h(x)$ is a monotonic , increasing and differentiable function of x such that (3.3) is a df.

Proof: First, we will prove (3.3) implies (3.2). We have (Athar *et al.*, 2003)

$$E[h(X_{U(i)}) \mid X_{U(m)} = x] = h(x) + \frac{i-m}{a}$$

for $F(x)=1-e^{-ah(x)}, x\in(\alpha,\beta); a>0$.

Therefore,

$$\sum_{i=r}^{s} b_i E[h(X_{U(i)}) \mid X_{U(m)}=x]=\sum_{i=r}^{s} b_i\left[h(x)+\frac{i-m}{a}\right]$$

$$=\frac{1}{a}\sum_{i=r}^{s} i b_i \ , \text{ as } \sum_{i=r}^{s} b_i=0. \tag{3.4}$$

Hence the 'if' part.

To prove the sufficiency part, we have

$$\sum_{i=r}^{s} b_i E[h(X_{U(i)}) \mid X_{U(m)}=x]=\frac{1}{a}\sum_{i=r}^{s} i b_i \tag{3.5}$$

or,

$$\sum_{i=r}^{s} b_i \frac{a}{\Gamma(i-m)}\int_x^{\beta} h(y)[-\ln\bar{F}(y)+\ln\bar{F}(x)]^{i-m-1}\frac{f(y)}{\bar{F}(x)}dy$$

$$=\sum_{i=r}^{s} i b_i . \tag{3.6}$$

Integrating left hand side of (3.6) by parts, we get

$$\sum_{i=r}^{s} b_i \frac{a}{\Gamma(i-m)}\int_x^{\beta} h'(y)[-\ln\bar{F}(y)+\ln\bar{F}(x)]^{i-m-1}\frac{\bar{F}(y)}{\bar{F}(x)}dy$$

$$+\sum_{i=r}^{s} b_i \frac{a}{\Gamma(i-m-1)}\int_x^{\beta} h(y)[-\ln\bar{F}(y)+\ln\bar{F}(x)]^{i-m-2}$$

$$\times\frac{f(y)}{\bar{F}(x)}dy=\sum_{i=r}^{s} i b_i . \tag{3.7}$$

That is,

$$\sum_{i=r}^{s} b_i \frac{a}{\Gamma(i-m)}\int_x^{\beta} h'(y)[-\ln\bar{F}(y)+\ln\bar{F}(x)]^{i-m-1}\frac{\bar{F}(y)}{\bar{F}(x)}dy$$

$$=\sum_{i=r}^{s} b_i(i-i+1)=0. \tag{3.8}$$

Now from (3.1), we have

$$\frac{1}{\Gamma(i-m)}\int_x^{\beta}[-\ln\bar{F}(y)+\ln\bar{F}(x)]^{i-m-1}\frac{f(y)}{\bar{F}(x)}dy=1.$$

Therefore,

$$\sum_{i=r}^{s} b_i\frac{1}{\Gamma(i-m)}\int_x^{\beta}[-\ln\bar{F}(y)+\ln\bar{F}(x)]^{i-m-1}\frac{f(y)}{\bar{F}(x)}dy=0. \tag{3.9}$$

Comparing (3.8) and (3.9), we get

$$ah'(y)\bar{F}(y)=f(y)$$

Implying

$$F(y)=1-\exp[-ah(y)];\ \ a>0$$

and hence the Theorem.

Remark 3.1: At $b_{r+1}=1$, $b_r=-1$ and $b_{r+2}=1$, $b_r=-1$ in Theorem 3.1, we get the results for exponential distribution as obtained by Lee (2001).

Remark 3.2: At $b_{r+3}=1$, $b_r=-1$ and $b_{r+4}=1$, $b_r=-1$ in Theorem 3.1, we get the results for exponential distribution as obtained by Lee *et al.* (2002).

Remark 3.3: Putting $b_s=1$ and $b_r=-1$ in Theorem 3.1, we get characterizing results as obtained by Khan *et al.* (2010b).

With the proper choice of a and $h(x)$, we get the characterizing results for distributions as given in Table 2.1.

4. Characterization theorems based on generalized order statistics

Here, we will consider two cases:

Case I: $m_1=m_2=\ldots=m_{n-1}=m$.

Case II: $\gamma_i\neq\gamma_j$, $i\neq j$, $i,j=1,\ldots,n-1$.

4.1. Characterization of distributions when $m_1 = m_2 = \ldots = m_{n-1} = m$.

The conditional *pdf* of $X(s,n,m,k)$ given $X(r,n,m,k) = x$, $1 \le r < s \le n$, is

$$f_{X(s,n,m,k)|X(r,n,m,k)}(y|x) = \frac{c_{s-1}}{(s-r-1)!\,c_{r-1}}$$

$$\times \frac{[h_m(F(y)) - h_m(F(x))]^{s-r-1}[1-F(y)]^{\gamma_s - 1} f(y)}{[1-F(x)]^{\gamma_{r+1}}}, \qquad (4.1)$$

where

$$c_{r-1} = \prod_{i=1}^{r} \gamma_i, \quad \gamma_i = k + (n-i)(m+1).$$

Theorem 4.1: Let $X(r,n,m,k)$, $r = 1,2,\ldots,n$ be the r^{th}- generalized order statistic from a continuous population with the *df* $F(x)$ and the *pdf* $f(x)$ over the support (α,β), where α and β may be finite or infinite. Then for $1 \le r < s < t \le n$,

$$\sum_{i=s}^{t} b_i E[h\{X(l,n,m,k)\} | X(r,n,m,k) = x] = \frac{1}{a}\sum_{i=s}^{t} b_l \sum_{j=r+1}^{i} \frac{1}{\gamma_j},$$

$$l = i-1, i \qquad (4.2)$$

if and only if

$$F(x) = 1 - e^{-ah(x)}, \quad x \in (\alpha,\beta), \quad a > 0, \qquad (4.3)$$

where b_i are real numbers $s \le i \le t$, satisfying $\sum_{i=s}^{t} b_i = 0$, for all $b_i \ne 0$ and $h(x)$ is a monotonic , increasing and differentiable function of x such that (4.3) is a *df* .

Proof: First, we will prove (4.3) implies (4.2). We have (Khan and Alzaid, 2004)

$$E[h\{X(i,n,m,k)\} \mid X(r,n,m,k)=x] = h(x) + \frac{1}{a}\sum_{j=r+1}^{i}\frac{1}{\gamma_j}$$

for $F(x)=1-e^{-ah(x)}, x\in(\alpha,\beta), a>0$.

Therefore,

$$\sum_{i=s}^{t} b_i E[h\{X(i,n,m,k)\} \mid X(r,n,m,k)=x]$$

$$=\sum_{i=s}^{t} b_i\left[h(x)+\frac{1}{a}\sum_{j=r+1}^{i}\frac{1}{\gamma_j}\right]$$

$$=\frac{1}{a}\sum_{i=s}^{t} b_i \sum_{j=r+1}^{i}\frac{1}{\gamma_j}, \text{ as } \sum_{i=s}^{t} b_i = 0.$$

Hence the 'if' part.

For the sufficiency part, we have

$$\sum_{i=s}^{t} b_i E[h\{X(i,n,m,k)\} \mid X(r,n,m,k)=x] = d\,, \tag{4.4}$$

where

$$d=\frac{1}{a}\sum_{i=s}^{t} b_i \sum_{j=r+1}^{i}\frac{1}{\gamma_j}$$

$$\sum_{i=s}^{t} b_i \frac{c_{i-1}}{(i-r-1)!c_{r-1}}\int_x^{\beta} h(y)\frac{[h_m(F(y))-h_m(F(x))]^{i-r-1}[\overline{F}(y)]^{\gamma_i\;-1}}{[\overline{F}(x)]^{\gamma_{r+1}}}$$

$$\times f(y)dy = d$$

$$\sum_{i=s}^{t} b_i \frac{c_{i-1}}{(i-r-1)!c_{r-1}}\int_x^{\beta} h(y)[h_m(F(y))-h_m(F(x))]^{i-r-1}$$

$$\times[\overline{F}(y)]^{\gamma_i-1} f(y)dy = d[\overline{F}(x)]^{\gamma_{r+1}}\,. \tag{4.5}$$

Integrating left hand side of (4.5) by parts, we get

$$\sum_{i=s}^{t} b_i \frac{c_{i-2}}{(i-r-2)!c_{r-1}} \int_x^\beta h(y)[h_m(F(y)) - h_m(F(x))]^{i-r-2}$$

$$[\bar{F}(y)]^{\gamma_{i-1}-1} f(y)dy + \sum_{i=s}^{t} b_i \frac{c_{i-2}}{(i-r-1)!c_{r-1}} \int_x^\beta h'(y)$$

$$\times [h_m(F(y)) - h_m(F(x))]^{i-r-1} [\bar{F}(y)]^{\gamma_i} dy = d[\bar{F}(x)]^{\gamma_{r+1}} \tag{4.6}$$

$$\sum_{i=s}^{t} b_i \frac{c_{i-2}}{(i-r-1)!c_{r-1}} \int_x^\beta h'(y)[h_m(F(y)) - h_m(F(x))]^{i-r-1}$$

$$\times [\bar{F}(y)]^{\gamma_i} dy = (d - d_1)[\bar{F}(x)]^{\gamma_{r+1}}, \tag{4.7}$$

where

$$d_1 = \frac{1}{a} \sum_{i=s}^{t} b_i \sum_{j=r+1}^{i-1} \frac{1}{\gamma_j}.$$

Differentiating $(i-r)$ times both the sides of (4.7) $w.r.t.\ x$, we get

$$\sum_{i=s}^{t} \frac{b_i}{\gamma_i} \frac{c_{i-1}}{c_{r-1}} h'(x)\bar{F}(x) = \sum_{i=s}^{t} \frac{b_i}{\gamma_i} \frac{c_{i-1}}{c_{r-1}} \frac{f(x)}{a}$$

$$\sum_{i=s}^{t} \frac{b_i}{\gamma_i} \frac{c_{i-1}}{c_{r-1}} \left[h'(x)\bar{F}(x) - \frac{f(x)}{a} \right] = 0$$

$$h'(x)\bar{F}(x) - \frac{f(x)}{a} = 0, \text{ as } \sum_{i=s}^{t} \frac{b_i}{\gamma_i} \frac{c_{i-1}}{c_{r-1}} \neq 0$$

or,

$$h'(x)\bar{F}(x) = \frac{f(x)}{a}$$

and hence the Theorem

$$F(x) = 1 - \exp[-ah(x)], \quad a > 0.$$

Remark 4.1: Putting $b_t = 1$ and $b_s = -1$ in Theorem 4.1, we get the result as obtained by Khan *et al.* (2011).

Remark 4.2: At $m = 0$, $k = 1$, Theorem 4.1 reduces for ordinary order statistics as given in Theorem 2.1.

Remark 4.3: At $m=-1, k=1$, Theorem 4.1 reduces for record statistics as given in Theorem 3.1.

4.2. Characterization of distributions when $\gamma_i \neq \gamma_j;\ i \neq j, i, j = 1, \ldots, n-1$.

The conditional *pdf* of $X(s,n,\tilde{m},k)$ given $X(r,n,\tilde{m},k)=x$, $1 \le r < s \le n$, is [Kamps and Cramer, 2001]:

$$f_{s|r}(y|x) = \frac{c_{s-1}}{c_{r-1}} \sum_{i=r+1}^{s} a_i^{(r)}(s) \frac{[\bar{F}(y)]^{\gamma_i - 1}}{[\bar{F}(x)]^{\gamma_i}} f(y), \quad x \le y, \tag{4.8}$$

where

$$\gamma_i = k + n - i + M_i, \quad M_i = \sum_{j=i}^{n-1} m_j \tag{4.9}$$

$$a_i(r) = \prod_{\substack{j=1 \\ j \neq i}}^{r} \frac{1}{(\gamma_j - \gamma_i)} \qquad , \gamma_i \neq \gamma_j,\ 1 \le i \le r \le n \tag{4.10}$$

$$a_i^{(r)}(s) = \prod_{\substack{j=r+1 \\ j \neq i}}^{s} \frac{1}{(\gamma_j - \gamma_i)},\ \gamma_i \neq \gamma_j,\ r+1 \le i \le s \le n \tag{4.11}$$

Theorem 4.2: Let $X(r,n,\tilde{m},k)$, $r=1,2,\ldots,n$ be the r^{th}- generalized order statistic from a continuous population with the *df* $F(x)$ and the *pdf* $f(x)$ on the support (α,β), where α and β may be finite or infinite. Then for $1 \le r < s < t \le n$,

$$\sum_{l=s}^{t} b_l E[h\{X(l,n,\tilde{m},k)\} \mid X(r,n,\tilde{m},k)=x] = \frac{1}{a} \sum_{l=s}^{t} b_l \sum_{j=r+1}^{l} \frac{1}{\gamma_j} \tag{4.12}$$

if and only if

$$F(x) = 1 - e^{-ah(x)}, \quad x \in (\alpha,\beta);\ a > 0 \tag{4.13}$$

where b_l are real numbers $s \le l \le t$, satisfying $\sum_{l=s}^{t} b_l = 0$, for all $b_l \ne 0$ and $h(x)$ is a monotonic , increasing and differentiable function of x such that (4.13) is a df .

Proof: From the Theorem 4.1, the necessary part follows.

For the sufficiency part.

$$\sum_{l=s}^{t} b_l E[h\{X(l,n,\tilde{m},k)\} \mid X(r,n,\tilde{m},k) = x] = d\,, \tag{4.14}$$

where

$$d = \frac{1}{a} \sum_{l=s}^{t} b_l \sum_{j=r+1}^{l} \frac{1}{\gamma_j}.$$

Therefore,

$$\sum_{l=s}^{t} b_l \frac{c_{l-1}}{c_{r-1}} \sum_{i=r+1}^{l} a_i^{(r)}(l) \int_x^{\beta} h(y) \frac{[\bar{F}(y)]^{\gamma_i - 1}}{[\bar{F}(x)]^{\gamma_i}} f(y) dy = d\,. \tag{4.15}$$

Integrating left hand side of (4.15) by parts, we get

$$\sum_{l=s}^{t} b_l \frac{c_{l-1}}{c_{r-1}} \sum_{i=r+1}^{l} a_i^{(r)}(l) \int_x^{\beta} h'(y) \frac{[\bar{F}(y)]^{\gamma_i}}{[\bar{F}(x)]^{\gamma_i}} dy$$

$$- \sum_{l=s}^{t} b_l \frac{c_{l-1}}{c_{r-1}} \sum_{i=r+1}^{l} a_i^{(r)}(l)(\gamma_i - 1) \int_x^{\beta} h(y) \frac{[\bar{F}(y)]^{\gamma_i - 1}}{[\bar{F}(x)]^{\gamma_i}} f(y) dy = d\,.$$

Using the relation $c_l = \gamma_{r+1} c_{l-1}$ and $a_i^{(r+1)}(l) = (\gamma_{r+1} - \gamma_i) a_i^{(r)}(l)$, we get

$$\sum_{l=s}^{t} b_l \frac{c_{l-1}}{c_{r-1}} \sum_{i=r+1}^{l} a_i^{(r)}(l) \int_x^{\beta} a\, h'(y) \frac{[\bar{F}(y)]^{\gamma_i}}{[\bar{F}(x)]^{\gamma_i}} dy = 0\,, \text{ as } \sum_{l=s}^{t} b_l = 0\,. \tag{4.16}$$

Also from the (4.8)

$$\frac{c_{l-1}}{c_{r-1}}\sum_{i=r+1}^{l} a_i^{(r)}(l)\int_x^{\beta}\frac{[\overline{F}(y)]^{\gamma_i-1}}{[\overline{F}(x)]^{\gamma_i}}f(y)dy=1.$$

Therefore,

$$\sum_{l=s}^{t} b_l\frac{c_{l-1}}{c_{r-1}}\sum_{i=r+1}^{l} a_i^{(r)}(l)\int_x^{\beta}\frac{[\overline{F}(y)]^{\gamma_i-1}}{[\overline{F}(x)]^{\gamma_i}}f(y)dy=0. \tag{4.17}$$

Comparing (4.16) and (4.17), we get

$$ah'(y)\overline{F}(y)=f(y).$$

That is,

$$F(y)=1-\exp[-ah(y)];\ \ a>0$$

and hence the Theorem.

Remark 4.4: It may be seen that when $\gamma_i \neq \gamma_j$ but $m_i = m_j = m$, then [Khan *et al.*, 2006]:

$$a_i^{(r)}(s)=\frac{1}{(m+1)^{s-r-1}}(-1)^{s-i}\frac{1}{(i-r-1)!(s-i)!}$$

$$a_i\,(r)=\frac{1}{(m+1)^{r-1}}(-1)^{r-i}\frac{1}{(i-1)!\,(r-i)!},\ \ c_{r-1}=\prod_{i=1}^{r}\gamma_i$$

and consequently conditional *pdf* given in (4.8) will reduce to the conditional *pdf* given in (4.1). Thus the Theorem 4.2 will reduce to the Theorem 4.1.

<u>**CHAPTER-III**</u>

CHARACTERIZATION OF DISTRIBUTIONS BY CONDITIONAL EXPECTATION OF GENERALIZED ORDER STATISTICS

1. Introduction

$X(1,n,\tilde{m},k), X(2,n,\tilde{m},k),\ldots,X(r,n,\tilde{m},k)$, $r=1,2,\ldots,n$ are called the r^{th} - generalized order statistics(gos) if their joint probability density function is given by [Kamps, 1995]:

$$k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}[1-F(x_i)]^{m_i} f(x_i)\right)[1-F(x_n)]^{k-1} f(x_n) \tag{1.1}$$

for $F^{-1}(0)\le x_1 \le x_2 \le \cdots \le x_n \le F^{-1}(1)$.

We, in this chapter have characterized two general class of distribution:

(i) $\bar{F}(x)=[1-(m+1)x^{\alpha}]^{\frac{1}{m+1}}, 0\le x\le \beta; \alpha>0$, where $\beta=(m+1)^{-\frac{1}{\alpha}}$

and

(ii) $\bar{F}(x)=[1-(m+1)e^{\alpha x}]^{\frac{1}{m+1}}, -\infty < x \le \beta; \alpha>0$,

where

$$\beta=-\frac{1}{\alpha}\ln(m+1)$$

through conditional expectation of non-adjacent generalized order statistics and have deduced it to known results. Further, using Meijer's

Part of the results of this chapter are contained in Faizan *et al.* (2013) and Khan *et al.* (2012b).

G-function, conditional expectation of generalized order statistics based on two non-adjacent generalized order statistics has been used to characterize a general form of distribution.

2. Characterization of distributions when $m_1 = m_2 = \ldots = m_{n-1} = m$.

The conditional *pdf* of $X(s,n,m,k)$ given $X(r,n,m,k) = x$, $1 \le r < s \le n$, is

$$f_{s|r}(y \mid x) = \frac{c_{s-1}}{c_{r-1}(s-r-1)!(m+1)^{s-r-1}}\left[1-\left(\frac{\overline{F}(y)}{\overline{F}(x)}\right)^{m+1}\right]^{s-r-1}$$

$$\times\left[\frac{\overline{F}(y)}{\overline{F}(x)}\right]^{k+(m+1)(n-s)-1}\frac{f(y)}{\overline{F}(x)}, \quad x < y \tag{2.1}$$

where

$$c_{r-1} = \prod_{i=1}^{r} \gamma_i, \; \gamma_i = k + (n-i)(m+1).$$

Theorem 2.1: Let $X(r,n,m,k)$ $r = 1,2,\ldots,n$ be the r^{th}-gos from a continuous population with the *df* $F(x)$ and the *pdf* $f(x)$, then for $1 \le r < s \le n$,

$$E[X^{\alpha}(s,n,m,k) \mid X(l,n,m,k) = x] = a^{*}_{s|l}x^{\alpha} + b^{*}_{s|l},$$

$$l = r, r+1 \tag{2.2}$$

if and only if

$$\overline{F}(x) = [1-(m+1)x^{\alpha}]^{\frac{1}{m+1}}, \quad 0 \le x \le \beta;\ \alpha > 0, \tag{2.3}$$

where

$$\beta = (m+1)^{-\frac{1}{\alpha}}, \quad a^{*}_{s|r} = \frac{\gamma_s}{\gamma_r} \text{ and } b^{*}_{s|r} = \frac{1}{m+1}[1 - a^{*}_{s|r}].$$

Proof: We have,

$$E[X^{\alpha}(s,n,m,k) \mid X(r,n,m,k)=x]$$

$$=\frac{c_{s-1}}{c_{r-1}(s-r-1)!(m+1)^{s-r-1}}\int_x^{\beta} y^{\alpha}\left[1-\left(\frac{\overline{F}(y)}{\overline{F}(x)}\right)^{m+1}\right]^{s-r-1}$$

$$\times\left[\frac{\overline{F}(y)}{\overline{F}(x)}\right]^{k+(m+1)(n-s)-1}\frac{f(y)}{\overline{F}(x)}dy,\quad x<y<\beta \tag{2.4}$$

Setting $u=\left[\frac{\overline{F}(y)}{\overline{F}(x)}\right]^{m+1}$, it reduces to

$$E[X^{\alpha}(s,n,m,k) \mid X(r,n,m,k)=x]=\frac{c_{s-1}}{c_{r-1}(s-r-1)!(m+1)^{s-r+1}}$$

$$\times\int_0^1[1-u\{1-(m+1)x^{\alpha}\}](1-u)^{s-r-1}[u]^{\frac{k+(m+1)(n-s)}{m+1}-1}du$$

$$=\frac{c_{s-1}}{c_{r-1}(s-r-1)!(m+1)^{s-r+1}}\int_0^1[u]^{\frac{k+(m+1)(n-s)}{m+1}-1}[1-u]^{s-r-1}du$$

$$-\frac{c_{s-1}}{c_{r-1}(s-r-1)!(m+1)^{s-r+1}}[1-(m+1)x^{\alpha}]$$

$$\times\int_0^1[u]^{\frac{k+(m+1)(n-s)}{m+1}+1-1}[1-u]^{s-r-1}du$$

$$=\frac{c_{s-1}}{c_{r-1}(s-r-1)!(m+1)^{s-r+1}}B\left(\frac{\gamma_s}{m+1},s-r\right)$$

$$-\frac{c_{s-1}}{c_{r-1}(s-r-1)!(m+1)^{s-r+1}}[1-(m+1)x^{\alpha}]B\left(\frac{\gamma_s}{m+1}+1,s-r\right)$$

$$=\frac{1}{(m+1)}-\frac{1}{(m+1)}[1-(m+1)x^{\alpha}]\frac{\gamma_s}{\gamma_r}. \tag{2.5}$$

Thus,

$$E[X^{\alpha}(s,n,m,k) \mid X(r,n,m,k)=x]=a^{*}_{s|r}x^{\alpha}+b^{*}_{s|r}.$$

This proves the necessary part.

To prove the sufficiency part, we have [Khan *et al.*, 2006]:

If $\quad E[X^{\alpha}(s,n,m,k) \mid X(r,n,m,k)=x]=g_{s|r}(x)$

then

$$\frac{f(x)}{\overline{F}(x)}=-\frac{1}{\gamma_{r+1}}\frac{g'_{s|r}(x)}{[g_{s|r}(x)-g_{s|r+1}(x)]}$$

Now, since

$$g_{s|r}(x)-g_{s|r+1}(x)=(a^{*}_{s|r}-a^{*}_{s|r+1})\left[x^{\alpha}-\frac{1}{m+1}\right]$$

$$=\frac{\gamma_s}{\gamma_r\gamma_{r+1}}\left[(m+1)x^{\alpha}-1\right].$$

Thus,

$$\frac{f(x)}{\overline{F}(x)}=\frac{\alpha x^{\alpha-1}}{[(m+1)x^{\alpha}-1]}$$

implying that

$$\overline{F}(x)=[1-(m+1)x^{\alpha}]^{\frac{1}{m+1}},\ 0\le x\le\beta;\alpha>0$$

and hence the Theorem.

Remark 2.1: At $m=0$, $k=1$, Theorem 2.1 reduces for order statistics: For $1\le r<s\le n$,

$$E[X^{\alpha}_{s:n} \mid X_{r:n}=x]=a^{*}_{s|r}x^{\alpha}+b^{*}_{s|r}$$

if and only if

$$F(x) = x^{\alpha}, \; 0 < x < 1; \alpha > 0$$

where

$$a_{s|r}^{*} = \left(\frac{n-s+1}{n-r+1}\right) \text{ and } b_{s|r}^{*} = \left(\frac{s-r}{n-r+1}\right)$$

as obtained by Franco and Ruiz (1997), Dembińska and Wesolowski (1998), Khan and Abouammoh (2000) and Khan and Alzaid (2004).

Remark 2.2: At $m=-1, k=1$, Theorem 2.1 reduces for record statistics: For $1 \le r < s$,

$$E[X_{U(s)}^{\alpha} \mid X_{U(r)} = x] = x^{\alpha}, \text{ with } a_{s|r}^{*} = 1 \text{ and } b_{s|r}^{*} = 0$$

if and only if

$$\overline{F}(x) = e^{-x^{\alpha}}, \quad 0 < x < \infty; \quad \alpha > 0$$

as obtained by Franco and Ruiz (1997) and Athar *et al.* (2003).

Theorem 2.2: Let $X(r,n,m,k)$, $r = 1,2,\ldots,n$ be the r^{th}-*gos* from a continuous population with the *df* $F(x)$ and the *pdf* $f(x)$, then for $1 \le r < s < t \le n$,

$$E[X^{\alpha}(t,n,m,k) \mid X(r,n,m,k) = x]$$

$$= a_{t|s}^{*} E[X^{\alpha}(s,n,m,k) \mid X(r,n,m,k) = x] + b_{t|s}^{*} \tag{2.6}$$

if and only if

$$\overline{F}(x) = [1-(m+1)x^{\alpha}]^{\frac{1}{m+1}}, \quad 0 \le x \le \beta; \quad \alpha > 0 \tag{2.7}$$

where

$$\beta = (m+1)^{-\frac{1}{\alpha}}, \quad a_{t|s}^{*} = \frac{\gamma_t}{\gamma_s} \quad \text{and} \quad b_{t|s}^{*} = \frac{1}{m+1}[1 - a_{t|s}^{*}].$$

Proof: To prove the necessary part, we have in view of Theorem 2.1,

$$E[X^{\alpha}(t,n,m,k) \mid X(r,n,m,k)=x] = a^{*}_{t|r}x^{\alpha} + b^{*}_{t|r}$$

$$= a^{*}_{t|r}\left(x^{\alpha} - \frac{1}{(m+1)}\right) + \frac{1}{(m+1)}$$

$$= a^{*}_{t|s}a^{*}_{s|r}\left(x^{\alpha} - \frac{1}{(m+1)}\right) + \frac{1}{(m+1)}$$

$$= a^{*}_{t|s}\left[a^{*}_{s|r}\left(x^{\alpha} - \frac{1}{(m+1)}\right) + \frac{1}{(m+1)}\right] + \frac{1}{(m+1)} - \frac{1}{(m+1)}a^{*}_{t|s}$$

$$= a^{*}_{t|s}[a^{*}_{s|r}x^{\alpha} + b^{*}_{s|r}] + b^{*}_{t|s}$$

as $\quad a^{*}_{t|r} = \dfrac{\gamma_t}{\gamma_r} = \dfrac{\gamma_t}{\gamma_s} \times \dfrac{\gamma_s}{\gamma_r} = a^{*}_{t|s}a^{*}_{s|r}$ and $b^{*}_{t|r} = \dfrac{1}{m+1}[1 - a^{*}_{t|r}]$.

That is,

$$E[X^{\alpha}(t,n,m,k) \mid X(r,n,m,k)=x]$$

$$= a^{*}_{t|s}E[X^{\alpha}(s,n,m,k) \mid X(r,n,m,k)=x] + b^{*}_{t|s}.$$

For the sufficiency part, we have

$$\frac{c_{t-1}}{c_{r-1}(t-r-1)!(m+1)^{t-r-1}}$$

$$\times \int_x^{\beta} y^{\alpha}[\overline{F}(x)^{m+1} - \overline{F}(y)^{m+1}]^{t-r-1}[\overline{F}(y)]^{\gamma_t-1}f(y)dy$$

$$= a^{*}_{t|s}\frac{c_{s-1}}{c_{r-1}(s-r-1)!(m+1)^{s-r-1}}$$

$$\times \int_x^{\beta} y^{\alpha}[\overline{F}(x)^{m+1} - \overline{F}(y)^{m+1}]^{s-r-1}[\overline{F}(y)]^{\gamma_s-1}f(y)dy$$

$$+ b^{*}_{t|s}[\overline{F}(x)]^{\gamma_{r+1}} \tag{2.8}$$

Differentiating $(s-r)$ times both the sides of $(2.8)\, w.r.t. x$, we get

$$\frac{c_{t-1}}{c_{s-1}(t-s-1)!(m+1)^{t-s-1}}$$

$$\times \int_x^\beta \frac{1}{[\overline{F}(x)]^{\gamma_{s+1}}} y^\alpha [\overline{F}(x)^{m+1} - \overline{F}(y)^{m+1}]^{t-s-1} [\overline{F}(y)]^{\gamma_t - 1} f(y) dy$$

$$= a^*_{t|s} x^\alpha + b^*_{t|s}$$

or,

$$g_{t|s}(x) = a^*_{t|s} x^\alpha + b^*_{t|s}.$$

Using the result [Khan *et al.*,2006], we have

$$\frac{f(x)}{\overline{F}(x)} = \frac{\alpha x^{\alpha-1}}{[(m+1)x^\alpha - 1]}$$

which implies

$$\overline{F}(x) = [1-(m+1)x^\alpha]^{\frac{1}{m+1}}, \ 0 \le x \le \beta; \alpha > 0$$

and hence the Theorem.

Remark 2.3: At $r = s$, Theorem 2.2 reduces to Theorem 2.1.

Theorem 2.3: Under the conditions as given in Theorem 2.1 and for $1 \le r < s \le n$,

$$E[e^{\alpha X(s,n,m,k)} \mid X(l,n,m,k) = x] = a^*_{s|l} e^{\alpha x} + b^*_{s|l}, l = r, r+1 \quad (2.9)$$

if and only if

$$\overline{F}(x) = [1-(m+1)e^{\alpha x}]^{\frac{1}{m+1}}, \ -\infty < x \le \beta; \ \alpha > 0 \quad (2.10)$$

where

$$\beta = -\frac{1}{\alpha}\ln(m+1) \ , \ \ a^*_{s|r} = \frac{\gamma_s}{\gamma_r} \text{ and } b^*_{s|r} = \frac{1}{m+1}[1 - a^*_{s|r}].$$

Proof: We have,

$$E[e^{\alpha X(s,n,m,k)} \mid X(r,n,m,k)=x]$$

$$= \frac{c_{s-1}}{c_{r-1}(s-r-1)!(m+1)^{s-r-1}} \int_x^{\beta} e^{\alpha y}\left[1-\left(\frac{\overline{F}(y)}{\overline{F}(x)}\right)^{m+1}\right]^{s-r-1}$$

$$\times\left[\frac{\overline{F}(y)}{\overline{F}(x)}\right]^{k+(m+1)(n-s)-1} \frac{f(y)}{\overline{F}(x)}dy, \quad x<y<\beta. \tag{2.11}$$

Setting $u=\left[\frac{\overline{F}(y)}{\overline{F}(x)}\right]^{m+1}$, it reduces to

$$E[e^{\alpha X(s,n,m,k)} \mid X(r,n,m,k)=x] = \frac{c_{s-1}}{c_{r-1}(s-r-1)!(m+1)^{s-r+1}}$$

$$\times \int_0^1 [1-u\{1-(m+1)e^{\alpha x}\}](1-u)^{s-r-1}[u]^{\frac{k+(m+1)(n-s)}{m+1}-1} du$$

$$= \frac{c_{s-1}}{c_{r-1}(s-r-1)!(m+1)^{s-r+1}} \int_0^1 [u]^{\frac{k+(m+1)(n-s)}{m+1}-1}[1-u]^{s-r-1} du$$

$$- \frac{c_{s-1}}{c_{r-1}(s-r-1)!(m+1)^{s-r+1}}[1-(m+1)e^{\alpha x}]$$

$$\times \int_0^1 [u]^{\frac{k+(m+1)(n-s)}{m+1}+1-1}[1-u]^{s-r-1} du$$

$$= \frac{c_{s-1}}{c_{r-1}(s-r-1)!(m+1)^{s-r+1}} B\left(\frac{\gamma_s}{m+1}, s-r\right)$$

$$- \frac{c_{s-1}}{c_{r-1}(s-r-1)!(m+1)^{s-r+1}}[1-(m+1)e^{\alpha x}] B\left(\frac{\gamma_s}{m+1}+1, s-r\right)$$

$$= \frac{1}{(m+1)} - \frac{1}{(m+1)}[1-(m+1)e^{\alpha x}]\frac{\gamma_s}{\gamma_r}.$$

Thus,

$$E[e^{\alpha X(s,n,m,k)} | X(r,n,m,k) = x] = a^*_{s|r} e^{\alpha x} + b^*_{s|r}. \tag{2.12}$$

This proves the necessary part.

For the sufficiency part, using the result [Khan *et al.*, 2006], we have

$$\frac{f(x)}{\overline{F}(x)} = \frac{\alpha e^{\alpha x}}{[(m+1)e^{\alpha x} - 1]}$$

which implies

$$\overline{F}(x) = [1-(m+1)e^{\alpha x}]^{\frac{1}{m+1}}, -\infty < x \le \beta; \alpha > 0$$

and hence the Theorem.

Remark 2.4: At $m = 0, k = 1$, Theorem 2.3 reduces for order statistics: For $1 \le r < s \le n$,

$$E[e^{\alpha X_{s:n}} | X_{r:n} = x] = a^*_{s|r} e^{\alpha x} + b^*_{s|r}$$

if and only if

$$F(x) = e^{\alpha x}, \quad -\infty < x < 0; \quad \alpha > 0,$$

where

$$a^*_{s|r} = \left(\frac{n-s+1}{n-r+1}\right) \text{ and } b^*_{s|r} = \left(\frac{s-r}{n-r+1}\right)$$

as obtained by Franco and Ruiz (1997).

Remark 2.5: At $m = -1, k = 1$, Theorem 2.3 reduces for record statistics: For $1 \le r < s$,

$$E[e^{\alpha X_{U(s)}} | X_{U(r)} = x] = e^{\alpha x}, \text{ with } a^*_{s|r} = 1 \text{ and } b^*_{s|r} = 0$$

if and only if

$$\overline{F}(x) = e^{-e^{\alpha x}}, -\infty < x < \infty; \alpha > 0$$

as obtained by Franco and Ruiz (1997).

Theorem 2.4: Under the conditions as given in Theorem 2.2 and for $1 \le r < s < t \le n$,

$$E[e^{\alpha X(t,n,m,k)} \mid X(r,n,m,k) = x]$$

$$= a^*_{t|s} E[e^{\alpha X(s,n,m,k)} \mid X(r,n,m,k) = x] + b^*_{t|s} \tag{2.13}$$

if and only if

$$\overline{F}(x) = [1 - (m+1)e^{\alpha x}]^{\frac{1}{m+1}}, \quad -\infty < x \le \beta; \quad \alpha > 0 , \tag{2.14}$$

where

$$\beta = -\frac{1}{\alpha}\ln(m+1), \quad a^*_{t|s} = \frac{\gamma_t}{\gamma_s} \quad \text{and} \quad b^*_{t|s} = \frac{1}{m+1}[1 - a^*_{t|s}].$$

Proof: For the necessary part, we have

$$E[e^{\alpha X(t,n,m,k)} \mid X(r,n,m,k) = x] = a^*_{t|r} e^{\alpha x} + b^*_{t|r}$$

$$= a^*_{t|r}\left(e^{\alpha x} - \frac{1}{(m+1)}\right) + \frac{1}{(m+1)}$$

$$= a^*_{t|s} a^*_{s|r}\left(e^{\alpha x} - \frac{1}{(m+1)}\right) + \frac{1}{(m+1)}$$

$$= a^*_{t|s}\left[a^*_{s|r}\left(e^{\alpha x} - \frac{1}{(m+1)}\right) + \frac{1}{(m+1)}\right] + \frac{1}{(m+1)} - \frac{1}{(m+1)} a^*_{t|s}$$

$$= a^*_{t|s}[a^*_{s|r} e^{\alpha x} + b^*_{s|r}] + b^*_{t|s},$$

where

$$a^*_{t|r} = \frac{\gamma_t}{\gamma_s} \times \frac{\gamma_s}{\gamma_r} = a^*_{t|s} a^*_{s|r} \quad \text{and} \quad b^*_{t|r} = \frac{1}{m+1}[1 - a^*_{t|r}].$$

That is,

$$E[e^{\alpha X(t,n,m,k)} \mid X(r,n,m,k) = x]$$

$$= a^*_{t|s} E[e^{\alpha X(s,n,m,k)} \mid X(r,n,m,k) = x] + b^*_{t|s} .$$

Hence, the necessary part.

For the sufficiency part, we have

$$\frac{c_{t-1}}{c_{r-1}(t-r-1)!(m+1)^{t-r-1}}$$

$$\times \int_x^\beta e^{\alpha y}[\overline{F}(x)^{m+1} - \overline{F}(y)^{m+1}]^{t-r-1}[\overline{F}(y)]^{\gamma_t - 1} f(y)dy$$

$$= a_{t|s}^* \frac{c_{s-1}}{c_{r-1}(s-r-1)!(m+1)^{s-r-1}}$$

$$\times \int_x^\beta e^{\alpha y}\, [\overline{F}(x)^{m+1} - \overline{F}(y)^{m+1}]^{s-r-1}[\overline{F}(y)]^{\gamma_s - 1} f(y)dy$$

$$+ b_{t|s}^* [\overline{F}(x)]^{\gamma_{r+1}}. \tag{2.15}$$

Differentiating $(s-r)$ times both the sides of (2.15) $w.r.t. x$, we get

$$\frac{c_{t-1}}{c_{s-1}(t-s-1)!(m+1)^{t-s-1}} \int_x^\beta \frac{1}{[\overline{F}(x)]^{\gamma_{s+1}}} e^{\alpha y}$$

$$\times [\overline{F}(x)^{m+1} - \overline{F}(y)^{m+1}]^{t-s-1}[\overline{F}(y)]^{\gamma_t - 1} f(y)dy$$

$$= a_{t|s}^* e^{\alpha x} + b_{t|s}^*$$

or,

$$g_{t|s}(x) = a_{t|s}^* e^{\alpha x} + b_{t|s}^*.$$

Using the result [Khan *et al.*, 2006], we have

$$\frac{f(x)}{\overline{F}(x)} = \frac{\alpha e^{\alpha x}}{[(m+1)e^{\alpha x} - 1]}$$

which implies

$$\overline{F}(x) = [1-(m+1)e^{\alpha x}]^{\frac{1}{m+1}}, \quad -\infty < x \le \beta; \quad \alpha > 0$$

hence the Theorem.

Remark 2.6: At $r = s$, Theorem 2.4 reduces to Theorem 2.3.

3. Characterization of distributions when $\gamma_i \neq \gamma_j;\ i \neq j;i,j=1,\dots,n-1$.

The conditional *pdf* of $X(s,n,\tilde{m},k)$ given $X(r,n,\tilde{m},k)=x$, $1 \le r < s \le n$, is

$$f_{s|r}(y|x)=\frac{c_{s-1}}{c_{r-1}}\sum_{i=r+1}^{s} a_i^{(r)}(s)\frac{[\bar{F}(y)]^{\gamma_i-1}}{[\bar{F}(x)]^{\gamma_i}}f(y),\quad x<y, \tag{3.1}$$

where

$$\gamma_i = k+n-i+M_i,\ M_i=\sum_{j=i}^{n-1} m_j\ ,c_{r-1}=\prod_{i=1}^{r}\gamma_i \tag{3.2}$$

and

$$a_i^{(r)}(s)=\prod_{\substack{j=r+1\\ j\neq i}}^{s}\frac{1}{(\gamma_j-\gamma_i)},\ \gamma_i\neq\gamma_j\ r+1\le i\le s\le n, i\neq j. \tag{3.3}$$

Theorem 3.1: Under the conditions as given in Theorem 2.1 and for $1 \le r < s \le n$,

$$E[X^{\alpha}(s,n,\tilde{m},k)\,|\,X(l,n,\tilde{m},k)=x]=a^{*}_{s|l}x^{\alpha}+b^{*}_{s|l},\ l=r,r+1 \tag{3.4}$$

if and only if

$$\bar{F}(x)=[1-(m+1)x^{\alpha}]^{\frac{1}{m+1}},\quad 0\le x\le\beta;\ \ \alpha>0, \tag{3.5}$$

where

$$\beta=(m+1)^{-\frac{1}{\alpha}},\ \ a^{*}_{s|r}=\prod_{j=r+1}^{s}\frac{\gamma_j}{\gamma_j+m+1}\ \text{and}\ \ b^{*}_{s|r}=\frac{1}{m+1}[1-a^{*}_{s|r}].$$

Proof: We have,

$$E[X^{\alpha}(s,n,\widetilde{m},k) \mid X(r,n,\widetilde{m},k)=x]$$

$$=\frac{c_{s-1}}{c_{r-1}}\sum_{i=r+1}^{s}a_i^{(r)}(s)\int_x^{\beta}y^{\alpha}\left[\frac{\overline{F}(y)}{\overline{F}(x)}\right]^{\gamma_i-1}\frac{f(y)}{\overline{F}(x)}dy. \tag{3.6}$$

Setting $u=\left[\frac{\overline{F}(y)}{\overline{F}(x)}\right]^{m+1}$, it reduces to

$$E[X^{\alpha}(s,n,\widetilde{m},k) \mid X(r,n,\widetilde{m},k)=x]$$

$$=\frac{c_{s-1}}{c_{r-1}}\sum_{i=r+1}^{s}a_i^{(r)}(s)\frac{1}{(m+1)^2}\int_0^1[1-u\{1-(m+1)x^{\alpha}\}][u]^{\frac{\gamma_{i-m-1}}{m+1}}du$$

$$=\frac{c_{s-1}}{c_{r-1}}\sum_{i=r+1}^{s}a_i^{(r)}(s)\frac{1}{(m+1)^2}\int_0^1[u]^{\frac{\gamma_{i-m-1}}{m+1}}du$$

$$-\frac{c_{s-1}}{c_{r-1}}\sum_{i=r+1}^{s}a_i^{(r)}(s)\frac{1}{(m+1)^2}[1-(m+1)x^{\alpha}]\int_0^1[u]^{\frac{\gamma_{i-m-1}}{m+1}+1}du$$

$$=\frac{c_{s-1}}{c_{r-1}}\sum_{i=r+1}^{s}a_i^{(r)}(s)\frac{1}{(m+1)^2}\left[\frac{m+1}{\gamma_i}-\{1-(m+1)x^{\alpha}\}\frac{m+1}{\gamma_{i+m+1}}\right]$$

$$=\frac{c_{s-1}}{c_{r-1}}\sum_{i=r+1}^{s}\frac{a_i^{(r)}(s)}{\gamma_{i+m+1}}x^{\alpha}+\frac{1}{(m+1)}\frac{c_{s-1}}{c_{r-1}}\sum_{i=r+1}^{s}\frac{a_i^{(r)}(s)}{\gamma_i}$$

$$-\frac{1}{(m+1)}\frac{c_{s-1}}{c_{r-1}}\sum_{i=r+1}^{s}\frac{a_i^{(r)}(s)}{\gamma_{i+m+1}}$$

$$=\frac{c_{s-1}}{c_{r-1}}\sum_{i=r+1}^{s}\frac{a_i^{(r)}(s)}{\gamma_{i+m+1}}x^{\alpha}+\frac{1}{(m+1)}\left[1-\frac{c_{s-1}}{c_{r-1}}\sum_{i=r+1}^{s}\frac{a_i^{(r)}(s)}{\gamma_{i+m+1}}\right]$$

$$=a_{s|r}^{*}x^{\alpha}+b_{s|r}^{*}=g_{s|r}(x),$$

where

$$\frac{c_{s-1}}{c_{r-1}}\sum_{i=r+1}^{s}\frac{a_i^{(r)}(s)}{\gamma_{i+m+1}}=\frac{c_{s-1}}{c_{r-1}}\sum_{i=r+1}^{s}\frac{1}{\prod\limits_{\substack{j=r+1\\j\neq i}}^{s}(\gamma_j-\gamma_i)\gamma_i+m+1}$$

$$=\prod_{j=r+1}^{s}\frac{\gamma_j}{\gamma_j+m+1}=a_{s|r}^{*}$$

and $b_{s|r}^{*}=\frac{1}{m+1}[1-a_{s|r}^{*}]$.

Therefore,

$$E[X^{\alpha}(s,n,\widetilde{m},k)\,|\,X(r,n,\widetilde{m},k)=x]=a_{s|r}^{*}x^{\alpha}+b_{s|r}^{*}=g_{s|r}(x).$$

This proves the necessary part.

To prove the sufficiency part, we have [Khan *et al.*,2006]:

$$\frac{f(x)}{\overline{F}(x)}=-\frac{1}{\gamma_{r+1}}\frac{g'_{s|r}(x)}{[g_{s|r}(x)-g_{s|r+1}(x)]}=\frac{\alpha x^{\alpha-1}}{[(m+1)x^{\alpha}-1]}$$

which implies

$$\overline{F}(x)=[1-(m+1)x^{\alpha}]^{\frac{1}{m+1}},\quad 0\le x\le\beta;\ \alpha>0$$

and hence the Theorem.

Theorem 3.2: Under the conditions as given in Theorem 2.2 and for $1\le r<s<t\le n$,

$$E[X^{\alpha}(t,n,\widetilde{m},k)\,|\,X(r,n,\widetilde{m},k)=x]$$

$$=a_{t|s}^{*}E[X^{\alpha}(s,n,\widetilde{m},k)\,|\,X(r,n,\widetilde{m},k)=x]+b_{t|s}^{*}\tag{3.7}$$

if and only if

$$\overline{F}(x)=[1-(m+1)x^{\alpha}]^{\frac{1}{m+1}},\quad 0\le x\le\beta;\ \alpha>0,\tag{3.8}$$

where

$$\beta=(m+1)^{-\frac{1}{\alpha}},\ \ a^*_{t|s}=\prod_{j=s+1}^{t}\frac{\gamma_j}{\gamma_j+m+1}\ \text{ and }\ b^*_{t|s}=\frac{1}{m+1}[1-a^*_{t|s}].$$

Proof: For the necessary part, we have

$$E[X^\alpha(t,n,\tilde{m},k)\,|\,X(r,n,\tilde{m},k)=x]=a^*_{t|r}x^\alpha+b^*_{t|r}$$

$$=a^*_{t|r}\left(x^\alpha-\frac{1}{(m+1)}\right)+\frac{1}{(m+1)}$$

$$=a^*_{t|s}a^*_{s|r}\left(x^\alpha-\frac{1}{(m+1)}\right)+\frac{1}{(m+1)}$$

$$=a^*_{t|s}\left[a^*_{s|r}\left(x^\alpha-\frac{1}{(m+1)}\right)+\frac{1}{(m+1)}\right]+\frac{1}{(m+1)}-\frac{1}{(m+1)}a^*_{t|s}$$

$$=a^*_{t|s}[a^*_{s|r}x^\alpha+b^*_{s|r}]+b^*_{t|s},$$

where

$$a^*_{t|r}=a^*_{t|s}a^*_{s|r}\ \text{ and }\ b^*_{t|r}=\frac{1}{m+1}[1-a^*_{t|r}].$$

That is,

$$E[X^\alpha(t,n,\tilde{m},k)\,|\,X(r,n,\tilde{m},k)=x]$$

$$=a^*_{t|s}E[X^\alpha(s,n,\tilde{m},k)\,|\,X(r,n,\tilde{m},k)=x]+b^*_{t|s}.$$

For the sufficiency part, we have

$$\frac{c_{t-1}}{c_{r-1}}\sum_{i=r+1}^{t}a_i^{(r)}(t)\int_x^\beta y^\alpha\left[\frac{\bar{F}(y)}{\bar{F}(x)}\right]^{\gamma_i}\frac{f(y)}{[\bar{F}(y)]}dy$$

$$=a^*_{t|s}\frac{c_{s-1}}{c_{r-1}}\sum_{i=r+1}^{s}a_i^{(r)}(s)\int_x^\beta y^\alpha\left[\frac{\bar{F}(y)}{\bar{F}(x)}\right]^{\gamma_i}\frac{f(y)}{[\bar{F}(y)]}dy+b^*_{t|s}. \tag{3.9}$$

Differentiating both the sides of (3.9) *w.r.t.* x, we get

$$\frac{c_{t-1}}{c_{r-1}}\sum_{i=r+1}^{t}a_i^{(r)}(t)\left[-\frac{x^\alpha f(x)}{\bar{F}(x)}+\gamma_i\int_x^\beta\frac{y^\alpha[\bar{F}(y)]^{\gamma_i-1}}{[\bar{F}(x)]^{\gamma_i+1}}f(x)f(y)dy\right]$$

$$=a_{t|s}^{*}\frac{c_{s-1}}{c_{r-1}}\sum_{i=r+1}^{s}a_i^{(r)}(s)\left[-\frac{x^\alpha f(x)}{\bar{F}(x)}+\gamma_i\int_x^\beta\frac{y^\alpha[\bar{F}(y)]^{\gamma_i-1}}{[\bar{F}(x)]^{\gamma_i+1}}f(x)f(y)dy\right]$$

or,

$$\frac{f(x)}{\bar{F}(x)}\left[-\frac{c_{t-1}}{c_{r-1}}\sum_{i=r+1}^{t}a_i^{(r)}(t)x^\alpha+\frac{c_{t-1}}{c_{r-1}}\sum_{i=r+1}^{t}\gamma_i a_i^{(r)}(t)\right.$$

$$\left.\times\int_x^\beta\frac{y^\alpha[\bar{F}(y)]^{\gamma_i-1}f(y)}{[\bar{F}(x)]^{\gamma_i}}dy\right]$$

$$=a_{t|s}^{*}\frac{f(x)}{\bar{F}(x)}\left[-\frac{c_{s-1}}{c_{r-1}}\sum_{i=r+1}^{s}a_i^{(r)}(s)x^\alpha+\frac{c_{s-1}}{c_{r-1}}\sum_{i=r+1}^{s}\gamma_i a_i^{(r)}(s)\right.$$

$$\left.\times\int_x^\beta\frac{y^\alpha[\bar{F}(y)]^{\gamma_i-1}f(y)}{[\bar{F}(x)]^{\gamma_i}}dy\right].$$

After noting that $\sum_{i=r+1}^{s}a_i^{(r)}(s)=0$ [Khan *et al.*,2006], $c_r=\gamma_{r+1}c_{r-1}$ and $a_i^{(r+1)}(t)=(\gamma_{r+1}-\gamma_i)a_i^{(r)}(t)$, we get

$$\frac{\gamma_{r+1}c_{t-1}}{c_{r-1}}\sum_{i=r+1}^{t}a_i^{(r)}(t)\int_x^\beta\frac{y^\alpha[\bar{F}(y)]^{\gamma_i-1}f(y)}{[\bar{F}(x)]^{\gamma_i}}dy$$

$$-\frac{\gamma_{r+1}c_{t-1}}{c_r}\sum_{i=r+2}^{t}a_i^{(r+1)}(t)\int_x^\beta\frac{y^\alpha[\bar{F}(y)]^{\gamma_i-1}f(y)}{[\bar{F}(x)]^{\gamma_i}}dy$$

$$=a_{t|s}^{*}\left[\frac{\gamma_{r+1}c_{s-1}}{c_{r-1}}\sum_{i=r+1}^{s}a_i^{(r)}(s)\int_x^\beta\frac{y^\alpha[\bar{F}(y)]^{\gamma_i-1}f(y)}{[\bar{F}(x)]^{\gamma_i}}dy\right.$$

$$\left.-\frac{\gamma_{r+1}c_{s-1}}{c_r}\sum_{i=r+2}^{s}a_i^{(r+1)}(s)\int_x^\beta\frac{y^\alpha[\bar{F}(y)]^{\gamma_i-1}f(y)}{[\bar{F}(x)]^{\gamma_i}}dy\right].$$

That is,

$$\gamma_{r+1}[g_{t|r}(x) - g_{t|r+1}(x)] = a^{*}_{t|s}\gamma_{r+1}[g_{s|r}(x) - g_{s|r+1}(x)],$$

where

$$g_{s|r}(x) = E[X^{\alpha}(s,n,\tilde{m},k) \mid X(r,n,\tilde{m},k) = x]$$

or,

$$g_{t|r}(x) - a^{*}_{t|s} g_{s|r}(x) = g_{t|r+1}(x) - a^{*}_{t|s} g_{s|r+1}(x)$$

$$= \cdots = g_{t|s}(x) - a^{*}_{t|s}(x) g_{s|s}(x) = b^{*}_{t|s}.$$

Noting that $g_{s|s}(x) = x^{\alpha}$, we have

$$g_{t|s}(x) = a^{*}_{t|s} x^{\alpha} + b^{*}_{t|s}.$$

Using the result [Khan *et al.*, 2006], we get

$$\frac{f(x)}{\overline{F}(x)} = \frac{\alpha x^{\alpha-1}}{[(m+1)x^{\alpha} - 1]}$$

which implies

$$\overline{F}(x) = [1-(m+1)x^{\alpha}]^{\frac{1}{m+1}}, \quad 0 \le x \le \beta \; ; \; \alpha > 0$$

and hence the Theorem.

Theorem 3.3: Under the conditions as given in Theorem 2.1 and for $1 \le r < s \le n$,

$$E[e^{\alpha X(s,n,\tilde{m},k)} \mid X(l,n,\tilde{m},k) = x] = a^{*}_{s|l} e^{\alpha x} + b^{*}_{s|l}, l = r, r+1 \qquad (3.10)$$

if and only if

$$\overline{F}(x) = [1-(m+1)e^{\alpha x}]^{\frac{1}{m+1}}, \quad -\infty < x \le \beta; \quad \alpha > 0, \qquad (3.11)$$

where

$$\beta = -\frac{1}{\alpha}\ln(m+1),\ \ a^{*}_{s|r} = \prod_{j=r+1}^{s} \frac{\gamma_j}{\gamma_j + m + 1} \ \text{ and } \ b^{*}_{s|r} = \frac{1}{m+1}[1 - a^{*}_{s|r}].$$

Proof: We have,

$$E[e^{\alpha X(s,n,\tilde{m},k)} \mid X(r,n,\tilde{m},k) = x]$$

$$= \frac{c_{s-1}}{c_{r-1}} \sum_{i=r+1}^{s} a_i^{(r)}(s) \int_x^{\beta} e^{\alpha y} \left[\frac{\overline{F}(y)}{\overline{F}(x)}\right]^{\gamma_i - 1} \frac{f(y)}{\overline{F}(x)} dy. \tag{3.12}$$

Setting $u = \left[\frac{\overline{F}(y)}{\overline{F}(x)}\right]^{m+1}$, it reduces to

$$E[e^{\alpha X(s,n,\tilde{m},k)} \mid X(r,n,\tilde{m},k) = x]$$

$$= \frac{c_{s-1}}{c_{r-1}} \sum_{i=r+1}^{s} a_i^{(r)}(s) \frac{1}{(m+1)^2} \int_0^1 [1 - u\{1 - (m+1)e^{\alpha x}\}][u]^{\frac{\gamma_{i-m-1}}{m+1}} du$$

$$= \frac{c_{s-1}}{c_{r-1}} \sum_{i=r+1}^{s} a_i^{(r)}(s) \frac{1}{(m+1)^2} \int_0^1 [u]^{\frac{\gamma_{i-m-1}}{m+1}} du$$

$$- \frac{c_{s-1}}{c_{r-1}} \sum_{i=r+1}^{s} a_i^{(r)}(s) \frac{1}{(m+1)^2} [1 - (m+1)e^{\alpha x}] \int_0^1 [u]^{\frac{\gamma_{i-m-1}}{m+1}+1} du$$

$$= \frac{c_{s-1}}{c_{r-1}} \sum_{i=r+1}^{s} a_i^{(r)}(s) \frac{1}{(m+1)^2} \left[\frac{m+1}{\gamma_i} - \{1 - (m+1)e^{\alpha x}\} \frac{m+1}{\gamma_{i+m+1}}\right]$$

$$= \frac{c_{s-1}}{c_{r-1}} \sum_{i=r+1}^{s} \frac{a_i^{(r)}(s)}{\gamma_{i+m+1}} e^{\alpha x} + \frac{1}{(m+1)} \frac{c_{s-1}}{c_{r-1}} \sum_{i=r+1}^{s} \frac{a_i^{(r)}(s)}{\gamma_i}$$

$$- \frac{1}{(m+1)} \frac{c_{s-1}}{c_{r-1}} \sum_{i=r+1}^{s} \frac{a_i^{(r)}(s)}{\gamma_{i+m+1}}$$

$$=\frac{c_{s-1}}{c_{r-1}}\sum_{i=r+1}^{s}\frac{a_i^{(r)}(s)}{\gamma_{i+m+1}}e^{\alpha x}+\frac{1}{(m+1)}\left[1-\frac{c_{s-1}}{c_{r-1}}\sum_{i=r+1}^{s}\frac{a_i^{(r)}(s)}{\gamma_{i+m+1}}\right]$$

$$=a_{s|r}^{*}e^{\alpha x}+b_{s|r}^{*}.$$

Therefore,

$$E[e^{\alpha X(s,n,\tilde{m},k)}\mid X(r,n,\tilde{m},k)=x]=a_{s|r}^{*}e^{\alpha x}+b_{s|r}^{*}=g_{s|r}(x)$$

and hence the necessary part.

To prove the sufficiency part, we have [Khan *et al.*,2006]:

$$\frac{f(x)}{\overline{F}(x)}=\frac{\alpha e^{\alpha x}}{[(m+1)e^{\alpha x}-1]}$$

which implies

$$\overline{F}(x)=[1-(m+1)e^{\alpha x}]^{\frac{1}{m+1}},\quad -\infty<x\le\beta;\ \ \alpha>0$$

and hence the Theorem.

Theorem 3.4: Under the conditions as given in Theorem 2.2 for $1\le r<s<t\le n$,

$$E[e^{\alpha X(t,n,\tilde{m},k)}\mid X(r,n,\tilde{m},k)=x]$$

$$=a_{t|s}^{*}E[e^{\alpha X(s,n,\tilde{m},k)}\mid X(r,n,\tilde{m},k)=x]+b_{t|s}^{*}\tag{3.13}$$

if and only if

$$\overline{F}(x)=[1-(m+1)e^{\alpha x}]^{\frac{1}{m+1}},\quad -\infty<x\le\beta;\ \ \alpha>0,\tag{3.14}$$

where

$$\beta=-\frac{1}{\alpha}\ln(m+1),\ \ a_{t|s}^{*}=\prod_{j=s+1}^{t}\frac{\gamma_j}{\gamma_j+m+1}\ \text{ and }\ b_{t|s}^{*}=\frac{1}{m+1}[1-a_{t|s}^{*}].$$

Proof: To prove the necessary part, we have in view of Theorem 3.3

$$E[e^{\alpha X(t,n,\tilde{m},k)} \mid X(r,n,\tilde{m},k)=x] = a^*_{t|r}e^{\alpha x} + b^*_{t|r}$$

$$= a^*_{t|r}\left(e^{\alpha x} - \frac{1}{(m+1)}\right) + \frac{1}{(m+1)}$$

$$= a^*_{t|s}a^*_{s|r}\left(e^{\alpha x} - \frac{1}{(m+1)}\right) + \frac{1}{(m+1)}$$

$$= a^*_{t|s}\left[a^*_{s|r}\left(e^{\alpha x} - \frac{1}{(m+1)}\right) + \frac{1}{(m+1)}\right] + \frac{1}{(m+1)} - \frac{1}{(m+1)}a^*_{t|s}$$

$$= a^*_{t|s}[a^*_{s|r}e^{\alpha x} + b^*_{s|r}] + b^*_{t|s},$$

where

$$a^*_{t|r} = \prod_{j=s+1}^{t}\frac{\gamma_j}{\gamma_j+m+1}\prod_{j=r+1}^{s}\frac{\gamma_j}{\gamma_j+m+1} = a^*_{t|s}a^*_{s|r}$$

and $b^*_{t|r} = \dfrac{1}{m+1}[1-a^*_{t|r}]$.

That is,

$$E[e^{\alpha X((t,n,\tilde{m},k)} \mid X(r,n,\tilde{m},k)=x]$$

$$= a^*_{t|s}E[e^{\alpha X(s,n,\tilde{m},k)} \mid X(r,n,\tilde{m},k)=x] + b^*_{t|s}.$$

This proves the necessary part.

For the sufficiency part, we have

$$\frac{c_{t-1}}{c_{r-1}}\sum_{i=r+1}^{t}a_i^{(r)}(t)\int_x^{\beta}e^{\alpha y}\left[\frac{\overline{F}(y)}{\overline{F}(x)}\right]^{\gamma_i-1}\frac{f(y)}{\overline{F}(x)}dy$$

$$= a^*_{t|s}\frac{c_{s-1}}{c_{r-1}}\sum_{i=r+1}^{s}a_i^{(r)}(s)\int_x^{\beta}e^{\alpha y}\left[\frac{\overline{F}(y)}{\overline{F}(x)}\right]^{\gamma_i-1}\frac{f(y)}{\overline{F}(x)}dy + b^*_{t|s}. \qquad (3.15)$$

Differentiating both the sides of (3.15) *w.r.t.x*, we get

$$\frac{c_{t-1}}{c_{r-1}}\sum_{i=r+1}^{t}a_i^{(r)}(t)\left[-\frac{e^{\alpha x}f(x)}{\overline{F}(x)}+\gamma_i\int_x^{\beta}\frac{e^{\alpha y}[\overline{F}(y)]^{\gamma_i-1}}{[\overline{F}(x)]^{\gamma_i+1}}f(x)f(y)dy\right]$$

$$=a_{t|s}^{*}\frac{c_{s-1}}{c_{r-1}}\sum_{i=r+1}^{s}a_i^{(r)}(s)\left[-\frac{e^{\alpha x}f(x)}{\overline{F}(x)}+\gamma_i\int_x^{\beta}\frac{e^{\alpha y}[\overline{F}(y)]^{\gamma_i-1}}{[\overline{F}(x)]^{\gamma_i+1}}f(x)f(y)dy\right]$$

or,

$$\frac{f(x)}{\overline{F}(x)}\left[-\frac{c_{t-1}}{c_{r-1}}\sum_{i=r+1}^{t}a_i^{(r)}(t)e^{\alpha x}+\frac{c_{t-1}}{c_{r-1}}\sum_{i=r+1}^{t}\gamma_i a_i^{(r)}(t)\right.$$
$$\left.\times\int_x^{\beta}\frac{e^{\alpha y}[\overline{F}(y)]^{\gamma_i-1}f(y)}{[\overline{F}(x)]^{\gamma_i}}dy\right]$$

$$=a_{t|s}^{*}\frac{f(x)}{\overline{F}(x)}\left[-\frac{c_{s-1}}{c_{r-1}}\sum_{i=r+1}^{s}a_i^{(r)}(s)e^{\alpha x}+\frac{c_{s-1}}{c_{r-1}}\sum_{i=r+1}^{s}\gamma_i a_i^{(r)}(s)\right.$$
$$\left.\times\int_x^{\beta}\frac{e^{\alpha y}[\overline{F}(y)]^{\gamma_i-1}f(y)}{[\overline{F}(x)]^{\gamma_i}}dy\right].$$

After noting that $\sum_{i=r+1}^{s}a_i^{(r)}(s)=0$ [Khan *et al.*, 2006], $c_r=\gamma_{r+1}c_{r-1}$ and $a_i^{(r+1)}(t)=(\gamma_{r+1}-\gamma_i)a_i^{(r)}(t)$, we get

$$\frac{\gamma_{r+1}c_{t-1}}{c_{r-1}}\sum_{i=r+1}^{t}a_i^{(r)}(t)\int_x^{\beta}\frac{e^{\alpha y}[\overline{F}(y)]^{\gamma_i-1}f(y)}{[\overline{F}(x)]^{\gamma_i}}dy$$
$$-\frac{\gamma_{r+1}c_{t-1}}{c_r}\sum_{i=r+2}^{t}a_i^{(r+1)}(t)\int_x^{\beta}\frac{e^{\alpha y}[\overline{F}(y)]^{\gamma_i-1}f(y)}{[\overline{F}(x)]^{\gamma_i}}dy$$

$$=a_{t|s}^{*}\left[\frac{\gamma_{r+1}c_{s-1}}{c_{r-1}}\sum_{i=r+1}^{s}a_i^{(r)}(s)\int_x^{\beta}\frac{e^{\alpha y}[\overline{F}(y)]^{\gamma_i-1}f(y)}{[\overline{F}(x)]^{\gamma_i}}dy\right.$$
$$\left.-\frac{\gamma_{r+1}c_{s-1}}{c_r}\sum_{i=r+2}^{s}a_i^{(r+1)}(s)\int_x^{\beta}\frac{e^{\alpha y}[\overline{F}(y)]^{\gamma_i-1}f(y)}{[\overline{F}(x)]^{\gamma_i}}dy\right].$$

That is,

$$\gamma_{r+1}[g_{t|r}(x) - g_{t|r+1}(x)] = a^*_{t|s}\gamma_{r+1}[g_{s|r}(x) - g_{s|r+1}(x)],$$

where

$$g_{s|r}(x) = E[e^{\alpha X(s,n,\tilde{m},k)} \mid X(r,n,\tilde{m},k) = x]$$

or,

$$g_{t|r}(x) - a^*_{t|s} g_{s|r}(x) = g_{t|r+1}(x) - a^*_{t|s} g_{s|r+1}(x)$$

$$= \cdots = g_{t|s}(x) - a^*_{t|s}(x) g_{s|s}(x) = b^*_{t|s}.$$

Noting that $g_{s|s}(x) = e^{\alpha x}$, we have

$$g_{t|s}(x) = E[e^{\alpha X(t,n,\tilde{m},k)} \mid X(s,n,\tilde{m},k) = x] = a^*_{t|s} e^{\alpha x} + b^*_{t|s}.$$

Using the result [Khan *et al.*,2006], we have

$$\frac{f(x)}{\overline{F}(x)} = \frac{\alpha e^{\alpha x}}{[(m+1)e^{\alpha x} - 1]}$$

which implies

$$\overline{F}(x) = [1 - (m+1)e^{\alpha x}]^{\frac{1}{m+1}}, \quad -\infty < x \le \beta; \quad \alpha > 0$$

and hence the Theorem.

Remark 3.1: It may be seen that when $\gamma_i \ne \gamma_j$ but $m_i = m_j = m$, then [Khan *et al.*, 2006]:

$$a_i^{(r)}(s) = \frac{1}{(m+1)^{s-r-1}(s-r-1)!}(-1)^{s-i}\binom{s-r-1}{s-i}$$

Therefore (3.1) reduces to

$$f_{s|r}(y\,|\,x)=\frac{c_{s-1}}{c_{r-1}(s-r-1)!(m+1)^{s-r-1}}$$

$$\times\sum_{i=r+1}^{s}(-1)^{s-i}\binom{s-r-1}{s-i}\left(\frac{\overline{F}(y)}{\overline{F}(x)}\right)^{k+(n-i)(m+1)-1}\frac{f(y)}{\overline{F}(x)}$$

$$=\frac{c_{s-1}}{c_{r-1}(s-r-1)!(m+1)^{s-r-1}}\sum_{i=r+1}^{s}(-1)^{s-i}\binom{s-r-1}{s-i}$$

$$\times\left[\left(\frac{\overline{F}(y)}{\overline{F}(x)}\right)\right]^{(m+1)(s-i)}\left[\left(\frac{\overline{F}(y)}{\overline{F}(x)}\right)\right]^{k+(m+1)(n-s)-1}\frac{f(y)}{\overline{F}(x)}$$

$$=\frac{c_{s-1}}{c_{r-1}(s-r-1)!(m+1)^{s-r-1}}$$

$$\times\left[1-\left(\frac{\overline{F}(y)}{\overline{F}(x)}\right)^{m+1}\right]^{s-r-1}\left[\left(\frac{\overline{F}(y)}{\overline{F}(x)}\right)\right]^{k+(m+1)(n-s)-1}\frac{f(y)}{\overline{F}(x)},\quad x<y$$

which is essentially (2.1).That is conditional *pdf* given in (3.1) reduces to the conditional *pdf* given in (2.1). Thus the results of Section 3 may be reduced to the results of Section 2.

4. Characterization of distributions using Meijer's G-function

The conditional P_F density function of $X(j,n,\widetilde{m},k)$ given $X(r,n,\widetilde{m},k)=x$ and $X(s,n,\widetilde{m},k)=y$, $1\le r<j<s\le n$, is given by

$$f_{j\,|\,r,s}(t|x,y)=\frac{1}{\overline{F}(t)}$$

$$\times\frac{G_{s-j}\left(\frac{\overline{F}(y)}{\overline{F}(t)}\middle|\gamma_{j+1},\ldots,\gamma_s\right)G_{j-r}\left(\frac{\overline{F}(t)}{\overline{F}(x)}\middle|\gamma_{r+1},\ldots,\gamma_j\right)}{G_{s-r}\left(\frac{\overline{F}(y)}{\overline{F}(x)}\middle|\gamma_{r+1},\ldots,\gamma_s\right)}I_{(x,y)}(t),\ (4.1)$$

where $G_{r,r}^{r,0}\left(x\middle|{\gamma_1,\ldots,\gamma_r \atop \gamma_1-1,\ldots,\gamma_r-1}\right)$ is as defined in Chapter-I.

Some results which are used in subsequent sections are reproduced here, [Mathai, 1993]:

A. $\lim\limits_{x\to 1-} G_r(x\,|\,\gamma_1,\ldots,\gamma_r)=\begin{cases}1, & r=1\\ 0, & r\geq 2\end{cases}$

B. $\dfrac{\partial}{\partial x}G_r(x\,|\,\gamma_1,\ldots,\gamma_r)=\dfrac{1}{x}[(\gamma_1-1)G_r(x\,|\,\gamma_1,\ldots,\gamma_r)-G_{r-1}(x\,|\,\gamma_2,\ldots,\gamma_r)]$

C. $\dfrac{\partial}{\partial x}G_r(x\,|\,\gamma_1,\ldots,\gamma_r)=\dfrac{1}{x}[(\gamma_1-1)G_r(x\,|\,\gamma_1,\ldots,\gamma_r)-G_{r-1}(x\,|\,\gamma_1,\ldots,\gamma_{r-1})]$

D. $x^a G_r(x\,|\,\gamma_1,\ldots,\gamma_r)=G_r(x\,|\,\gamma_1+a,\ldots,\gamma_r+a),\ \ a\in R.$

Theorem 4.1: Let $X(i,n,\tilde{m},k),\, i=1,\ldots,n$ be the $i^{th}-gos$ from a continuous population with the *df* $F(x)$ and the *pdf* $f(x)$ over the support (α,β) and $h(t)$ be a monotonic and differentiable function of t. If for two consecutive values r and $r+1$, $1<r+1<j<s\leq n$,

$$g_{j|l,s}(x,y)=E[h\{X(j,n,\tilde{m},k)\}\,|\,X(l,n,\tilde{m},k)=x,X(s,n,\tilde{m},k)=y],$$

$l=r,r+1$ exist, then

$$(\gamma_{r+1}-1)\frac{f(x)}{\overline{F}(x)}-\frac{\dfrac{\partial}{\partial x}G_{s-r}\left(\dfrac{\overline{F}(y)}{\overline{F}(x)}\middle|\gamma_{r+1},\ldots,\gamma_s\right)}{G_{s-r}\left(\dfrac{\overline{F}(y)}{\overline{F}(x)}\middle|\gamma_{r+1},\ldots,\gamma_s\right)}$$

$$=\frac{\dfrac{\partial}{\partial x}g_{j|r,s}(x,y)}{[g_{j|r,s}(x,y)-g_{j|r+1,s}(x,y)]} \tag{4.2}$$

and

$$\frac{G_{s-r}\left(\frac{\overline{F}(y)}{\overline{F}(x)}\middle|\gamma_{r+1}-\gamma_{r+1}+1,\ldots,\gamma_s-\gamma_{r+1}+1\right)}{G_{s-r}(\overline{F}(y)|\gamma_{r+1}-\gamma_{r+1}+1,\ldots,\gamma_s-\gamma_{r+1}+1)}$$

$$=\exp\left[-\int_{\alpha}^{x}D_1(t,y)dt\right], \tag{4.3}$$

where $g(\ )$ is a finite and differentiable function of x, and

$$D_1(x,y)=\frac{\frac{\partial}{\partial x}g_{j|r,s}(x,y)}{[g_{j|r,s}(x,y)-g_{j|r+1,s}(x,y)]}. \tag{4.4}$$

Proof: We have,

$$g_{j|r,s}(x,y)$$

$$=\int_x^y \frac{h(t)}{\overline{F}(t)}\frac{G_{s-j}\left(\frac{\overline{F}(y)}{\overline{F}(t)}\middle|\gamma_{j+1},\ldots,\gamma_s\right)G_{j-r}\left(\frac{\overline{F}(t)}{\overline{F}(x)}\middle|\gamma_{r+1},\ldots,\gamma_j\right)}{G_{s-r}\left(\frac{\overline{F}(y)}{\overline{F}(x)}\middle|\gamma_{r+1},\ldots,\gamma_s\right)}f(t)dt$$

or,

$$g_{j|r,s}(x,y)G_{s-r}\left(\frac{\overline{F}(y)}{\overline{F}(x)}\middle|\gamma_{r+1},\ldots,\gamma_s\right)$$

$$=\int_x^y \frac{h(t)}{\overline{F}(t)}G_{s-j}\left(\frac{\overline{F}(y)}{\overline{F}(t)}\middle|\gamma_{j+1},\ldots,\gamma_s\right)G_{j-r}\left(\frac{\overline{F}(t)}{\overline{F}(x)}\middle|\gamma_{r+1},\ldots,\gamma_j\right)f(t)dt \tag{4.5}$$

Differentiating both the sides of (4.5) *w.r.t.* x,

$$\frac{\partial}{\partial x}g_{j|r,s}(x,y)G_{s-r}\left(\frac{\overline{F}(y)}{\overline{F}(x)}\middle|\gamma_{r+1},\ldots,\gamma_s\right)$$

$$+g_{j|r,s}(x,y)\left[\frac{\partial}{\partial x}G_{s-r}\left(\frac{\overline{F}(y)}{\overline{F}(x)}\middle|\gamma_{r+1},\ldots,\gamma_s\right)\right]$$

$$= \int_x^y \frac{h(t)}{\overline{F}(t)} G_{s-j}\left(\frac{\overline{F}(y)}{\overline{F}(t)} \middle| \gamma_{j+1},\ldots,\gamma_s \right) \left\{ \frac{\partial}{\partial x} G_{j-r}\left(\frac{\overline{F}(t)}{\overline{F}(x)} \middle| \gamma_{r+1},\ldots,\gamma_j \right) \right\} f(t)dt .$$

Using the relation (A and B), we have

$$\frac{\partial}{\partial x} g_{j|r,s}(x,y) G_{s-r}\left(\frac{\overline{F}(y)}{\overline{F}(x)} \middle| \gamma_{r+1},\ldots,\gamma_s \right)$$

$$+ g_{j|r,s}(x,y) \frac{f(x)}{\overline{F}(x)} \left[(\gamma_{r+1} - 1) G_{s-r}\left(\frac{\overline{F}(y)}{\overline{F}(x)} \middle| \gamma_{r+1},\ldots,\gamma_s \right) \right]$$

$$- g_{j|r,s}(x,y) \frac{f(x)}{\overline{F}(x)} \left[G_{s-r-1}\left(\frac{\overline{F}(y)}{\overline{F}(x)} \middle| \gamma_{r+2},\ldots,\gamma_s \right) \right]$$

$$= \frac{f(x)}{\overline{F}(x)} \int_x^y \frac{h(t)}{\overline{F}(t)} G_{s-j}\left(\frac{\overline{F}(y)}{\overline{F}(t)} \middle| \gamma_{j+1},\ldots,\gamma_s \right)$$

$$\times \left[(\gamma_{r+1} - 1) G_{j-r}\left(\frac{\overline{F}(t)}{\overline{F}(x)} \middle| \gamma_{r+1},\ldots,\gamma_j \right) - G_{j-r-1}\left(\frac{\overline{F}(t)}{\overline{F}(x)} \middle| \gamma_{r+2},\ldots,\gamma_j \right) \right] f(t)dt$$

or,

$$\frac{\partial}{\partial x} g_{j|r,s}(x,y) G_{s-r}\left(\frac{\overline{F}(y)}{\overline{F}(x)} \middle| \gamma_{r+1},\ldots,\gamma_s \right)$$

$$+ g_{j|r,s}(x,y) \frac{f(x)}{\overline{F}(x)} \left[(\gamma_{r+1} - 1) G_{s-r}\left(\frac{\overline{F}(y)}{\overline{F}(x)} \middle| \gamma_{r+1},\ldots,\gamma_s \right) \right.$$

$$\left. - G_{s-r-1}\left(\frac{\overline{F}(y)}{\overline{F}(x)} \middle| \gamma_{r+2},\ldots,\gamma_s \right) \right]$$

$$= \frac{f(x)}{\overline{F}(x)} \left[g_{j\,r,s}(x,y)(\gamma_{r+1} - 1) G_{s-r}\left(\frac{\overline{F}(y)}{\overline{F}(x)} \middle| \gamma_{r+1},\ldots,\gamma_s \right) \right.$$

$$\left. - g_{j|r+1,s}(x,y) G_{s-r-1}\left(\frac{\overline{F}(y)}{\overline{F}(x)} \middle| \gamma_{r+2},\ldots,\gamma_s \right) \right]$$

implying that

$$\frac{f(x)}{\overline{F}(x)} \frac{G_{s-r-1}\left(\frac{\overline{F}(y)}{\overline{F}(x)} | \gamma_{r+2}, \ldots, \gamma_s\right)}{G_{s-r}\left(\frac{\overline{F}(y)}{\overline{F}(x)} | \gamma_{r+1}, \ldots, \gamma_s\right)} = \frac{\frac{\partial}{\partial x} g_{j|r,s}(x,y)}{[g_{j|r,s}(x,y) - g_{j|r+1,s}(x,y)]}. \tag{4.6}$$

Now, in view of (B)

$$\frac{f(x)}{\overline{F}(x)} G_{s-r-1}\left(\frac{\overline{F}(y)}{\overline{F}(x)} | \gamma_{r+2}, \ldots, \gamma_s\right) = -\frac{\partial}{\partial x} G_{s-r}\left(\frac{\overline{F}(y)}{\overline{F}(x)} | \gamma_{r+1}, \ldots, \gamma_s\right)$$
$$+ \frac{f(x)}{\overline{F}(x)} (\gamma_{r+1} - 1) G_{s-r}\left(\frac{\overline{F}(y)}{\overline{F}(x)} | \gamma_{r+1}, \ldots, \gamma_s\right)$$

we can therefore write (4.6) as,

$$(\gamma_{r+1} - 1) \frac{f(x)}{\overline{F}(x)} - \frac{\frac{\partial}{\partial x} G_{s-r}\left(\frac{\overline{F}(y)}{\overline{F}(x)} | \gamma_{r+1}, \ldots, \gamma_s\right)}{G_{s-r}\left(\frac{\overline{F}(y)}{\overline{F}(x)} | \gamma_{r+1}, \ldots, \gamma_s\right)}$$

$$= \frac{\frac{\partial}{\partial x} g_{j|r,s}(x,y)}{[g_{j|r,s}(x,y) - g_{j|r+1,s}(x,y)]} \tag{4.7}$$

and hence (4.2) is proved.

Integrating both the sides of (4.7) *w.r.t.* x over (α, x) and using the relation (D) we get,

$$(\gamma_{r+1} - 1) \log[\overline{F}(t)] + \log G_{s-r}\left(\frac{\overline{F}(y)}{\overline{F}(t)} | \gamma_{r+1}, \ldots, \gamma_s\right)\Bigg|_{\alpha}^{x} = -\int_{\alpha}^{x} D_1(t,y)\,dt$$

or,

$$\left[\log\left\{[\overline{F}(y)]^{\gamma_{r+1}-1} \left[\frac{\overline{F}(y)}{\overline{F}(t)}\right]^{-\gamma_{r+1}+1} G_{s-r}\left(\frac{\overline{F}(y)}{\overline{F}(t)} | \gamma_{r+1}, \ldots, \gamma_s\right)\right\}\right]_{\alpha}^{x}$$

$$= -\int_{\alpha}^{x} D_1(t,y)\,dt$$

or,

$$\left[\log\left\{[\overline{F}(y)]^{\gamma_{r+1}-1} G_{s-r}\left(\frac{\overline{F}(y)}{\overline{F}(t)}\Big|\gamma_{r+1}-\gamma_{r+1}+1,\ldots,\gamma_s-\gamma_{r+1}+1\right)\right\}\right]_{\alpha}^{x}$$

$$= -\int_{\alpha}^{x} D_1(t,y)dt$$

$$\frac{G_{s-r}\left(\frac{\overline{F}(y)}{\overline{F}(x)}\Big|\gamma_{r+1}-\gamma_{r+1}+1,\ldots,\gamma_s-\gamma_{r+1}+1\right)}{G_{s-r}(\overline{F}(y)|\gamma_{r+1}-\gamma_{r+1}+1,\ldots,\gamma_s-\gamma_{r+1}+1)} = \exp\left[-\int_{\alpha}^{x} D_1(t,y)dt\right]$$

and hence the Theorem is proved.

Corollary 4.1:

$$\frac{[\overline{F}(x)]^{\gamma_{r+1}} B_r{}^s(x,y)}{B_r{}^s(\alpha,y)} = \exp\left[-\int_{\alpha}^{x} D_1(t,y)dt\right], \tag{4.8}$$

where

$$B_r{}^s(x,y) = \sum_{i=r+1}^{s} a_i^{(r)}(s)\left(\frac{\overline{F}(y)}{\overline{F}(x)}\right)^{\gamma_i}.$$

Proof: Using residue Theorem, it can be proved that

$$G_{s-r}\left(\frac{\overline{F}(y)}{\overline{F}(x)}\Big|\gamma_{r+1},\ldots,\gamma_s\right) = \left(\sum_{i=r+1}^{s} a_i^{(r)}(s)\left[\frac{\overline{F}(y)}{\overline{F}(x)}\right]^{\gamma_i-1}\right),$$

where

$$a_i^{(r)}(s) = \prod_{\substack{j=r+1\\ j\neq i}}^{s} \frac{1}{(\gamma_j-\gamma_i)}, \quad \gamma_j \neq \gamma_i,\ r+1 \le i \le s \le n.$$

Therefore,

$$\frac{G_{s-r}\left(\frac{\overline{F}(y)}{\overline{F}(x)}\Big|\gamma_{r+1}-\gamma_{r+1}+1,\ldots,\gamma_s-\gamma_{r+1}+1\right)}{G_{s-r}(\overline{F}(y)|\gamma_{r+1}-\gamma_{r+1}+1,\ldots,\gamma_s-\gamma_{r+1}+1)}$$

or,

$$=\frac{\left[\frac{\overline{F}(y)}{\overline{F}(x)}\right]^{-\gamma_{r+1}+1} G_{s-r}\left(\frac{\overline{F}(y)}{\overline{F}(x)}\Big|\gamma_{r+1},\ldots,\gamma_s\right)}{[\overline{F}(y)]^{-\gamma_{r+1}+1} G_{s-r}(\overline{F}(y)|\gamma_{r+1},\ldots,\gamma_s)}$$

$$=[\overline{F}(x)]^{\gamma_{r+1}} \frac{\sum_{i=r+1}^{s} a_i^{(r)}(s)\left[\frac{\overline{F}(y)}{\overline{F}(x)}\right]^{\gamma_i}\left[\frac{\overline{F}(x)}{\overline{F}(y)}\right]}{\sum_{i=r+1}^{s} a_i^{(r)}(s)[\overline{F}(y)]^{\gamma_i}[\overline{F}(y)]^{-1}}.$$

Thus,

$$\frac{[\overline{F}(x)]^{\gamma_{r+1}} B_r^{\,s}(x,y)}{B_r^{\,s}(\alpha,y)} = \exp\left[-\int_{\alpha}^{x} D_1(t,y)dt\right] \tag{4.9}$$

as obtained by Ahsanullah *et al.*(2009).

Also since for $m_1 = \ldots = m_{n-1} = m \neq -1$,

$$a_i^{(r)}(s) = \frac{1}{\prod_{\substack{j=r+1 \\ i\neq j}}^{s} (\gamma_j - \gamma_i)} = \frac{(-1)^{s-i}}{(m+1)^{s-r-1}(s-r-1)!}\binom{s-r-1}{s-i}.$$

Thus the LHS of (4.9) for $m_1 = \ldots = m_{n-1} = m \neq -1$,reduces to

$$\frac{[\overline{F}(x)]^{\gamma_{r+1}} \mathrm{B}_r^s(x,y)}{\mathrm{B}_r^s(\alpha,y)} = \frac{[\overline{F}(x)]^{\gamma_{r+1}}\left(\frac{\overline{F}(y)}{\overline{F}(x)}\right)^{\gamma_s} \frac{1}{(m+1)^{s-r-1}(s-r-1)!}}{[\overline{F}(y)]^{\gamma_s} \frac{1}{(m+1)^{s-r-1}(s-r-1)!}}$$

$$\times \frac{\sum_{i=r+1}^{s} (-1)^{s-i} \frac{(s-r-1)!}{(i-r-1)!(s-i)!}\left(\frac{\overline{F}(y)}{\overline{F}(x)}\right)^{\gamma_i-\gamma_s}}{\sum_{i=r+1}^{s} (-1)^{s-i} \frac{(s-r-1)!}{(i-r-1)!(s-i)!}[\overline{F}(y)]^{\gamma_i-\gamma_s}}$$

$$= \frac{[\overline{F}(x)^{m+1}]^{(s-r-1)}\left[1-\frac{\overline{F}(y)^{m+1}}{\overline{F}(x)^{m+1}}\right]^{(s-r-1)}}{[1-\overline{F}(y)^{m+1}]^{(s-r-1)}}$$

$$\times \frac{[\{1-\overline{F}(y)^{m+1}\}-\{1-\overline{F}(x)^{m+1}\}]^{(s-r-1)}}{[1-\overline{F}(y)^{m+1}]^{(s-r-1)}}$$

$$= \exp\left[-\int_{\alpha}^{x} D_1(t,y)dt\right]$$

implying that

$$\frac{1-[\overline{F}(x)]^{m+1}}{1-[\overline{F}(y)]^{m+1}} = 1-\exp\left[-\frac{1}{(s-r-1)}\int_{\alpha}^{x} D_1(t,y)dt\right], \quad m>-1 \qquad (4.10)$$

and

$$\frac{\log \overline{F}(x)}{\log \overline{F}(y)} = 1-\exp\left[-\frac{1}{(s-r-1)}\int_{\alpha}^{x} D_1(t,y)dt\right], \quad m=-1 \qquad (4.11)$$

as obtained by Ahsanullah *et al.* (2009).

Remark 4.1: At $\gamma_s = 0$, i.e. at $s = k+n+M_s$, by convention $X(s,n,\tilde{m},k) = y = \beta$ and hence $\overline{F}(y) = 0$. Therefore,

$$\overline{F}(x) = \exp\left(-\frac{1}{\gamma_{r+1}}\int_{\alpha}^{x} \frac{g'_{j|r}(t)}{[g_{j|r}(t)-g_{j|r+1}(t)]}dt\right), \qquad (4.12)$$

where

$$g_{j|r}(x) = E[\{X(j,n,\tilde{m},k)\} \mid X(r,n,\tilde{m},k) = x]$$

as given by Khan *et al.* (2006) and Bieniek (2009).

For order statistics, Theorem 4.1 reduces as,

$$\frac{F(x)}{F(y)} = 1-\exp\left[-\frac{1}{(s-r-1)}\int_{\alpha}^{x} D_1(t,y)dt\right],$$

where $g_{j|r,s}(x,y)=E[h(X_{j:n})\,|\,X_{s:n}=x, X_{r:n}=y]$ and $D_1(t,y)$ is as defined in (4.4), as given by Khan *et al.* (2007).

At $s=n+1, X_{n+1:n}=y=\beta$ by convention and hence $\overline{F}(y)=0$. We have,

$$1-F(x)=\exp\left(-\frac{1}{(n-r)}\int_\alpha^x D_1(t)dt\right),$$

where

$$g_{j|r}(x)=E[h(X_{j:n})\,|\,X_{r:n}=x]$$

and $$D_1(x)=\frac{g'_{j|r}(x)}{[g_{j|r}(x)-g_{j|r+1}(x)]}$$

as obtained by Khan *et al.* (2006) and Beg and Ahsanullah (2006).

Theorem 4.2: Let $X(i,n,\tilde{m},k)$, $i=1,\ldots,n$ be the $i^{th}-gos$ from a continuous population with the *df* $F(x)$ and the *pdf* $f(x)$ over the support (α,β) and $h(t)$ be a monotonic and differentiable function of t. If for two consecutive values $s-1$ and s, $1\le r<j<s-1<n$,

$$g_{j|r,l}(x,y)=E[h\{(X(j,n,\tilde{m},k)\}|X(r,n,\tilde{m},k)=x, X(l,n,\tilde{m},k)=y],$$

$$l=s-1,\, s$$

exist, then

$$(\gamma_s-1)\frac{f(y)}{\overline{F}(y)}+\frac{\frac{\partial}{\partial y}G_{s-r}\left(\frac{\overline{F}(y)}{\overline{F}(x)}|\gamma_{r+1},\ldots,\gamma_s\right)}{G_{s-r}\left(\frac{\overline{F}(y)}{\overline{F}(x)}|\gamma_{r+1},\ldots,\gamma_s\right)}$$

$$=\frac{\frac{\partial}{\partial y}g_{j|r,s}(x,y)}{[g_{j|r,s-1}(x,y)-g_{j|r,s}(x,y)]} \tag{4.13}$$

and

$$G_{s-r}\left(\frac{\overline{F}(y)}{\overline{F}(x)}\middle|\gamma_{r+1}-\gamma_s+1,\ldots,\gamma_s-\gamma_s+1\right)=a_s^{(r)}(s)\exp\left[-\int_y^{\beta}D_2(x,t)dt\right],$$

$$\forall\,\gamma_i>\gamma_s,\ i=r+1,\ldots,s-1 \qquad (4.14)$$

and for $\gamma_{r+1}=\ldots=\gamma_s$,

$$\frac{1+\log\overline{F}(y)}{1+\log\overline{F}(x)}=1-\exp\left[-\frac{1}{(s-r-1)}\int_y^{q}D_2(x,t)dt\right], \qquad (4.15)$$

where $\quad q\in(\alpha,\beta), \quad -\log\overline{F}(q)=1 \qquad (4.16)$

and

$$D_2(x,y)=\frac{\frac{\partial}{\partial y}g_{j|r,s}(x,y)}{[g_{j|r,s-1}(x,y)-g_{j|r,s}(x,y)]}. \qquad (4.17)$$

Proof: We have,

$$g_{j|r,s}(x,y)$$

$$=\int_x^y\frac{h(t)}{\overline{F}(t)}\frac{G_{s-j}\left(\frac{\overline{F}(y)}{\overline{F}(t)}\middle|\gamma_{j+1},\ldots,\gamma_s\right)G_{j-r}\left(\frac{\overline{F}(t)}{\overline{F}(x)}\middle|\gamma_{r+1},\ldots,\gamma_j\right)}{G_{s-r}\left(\frac{\overline{F}(y)}{\overline{F}(x)}\middle|\gamma_{r+1},\ldots,\gamma_s\right)}f(t)dt$$

or,

$$g_{j|r,s}(x,y)G_{s-r}\left(\frac{\overline{F}(y)}{\overline{F}(x)}\middle|\gamma_{r+1},\ldots,\gamma_s\right)=\int_x^y\frac{h(t)}{\overline{F}(t)}G_{s-j}$$

$$\times\left(\frac{\overline{F}(y)}{\overline{F}(t)}\middle|\gamma_{j+1},\ldots,\gamma_s\right)G_{j-r}\left(\frac{\overline{F}(t)}{\overline{F}(x)}\middle|\gamma_{r+1},\ldots,\gamma_j\right)f(t)dt. \qquad (4.18)$$

Differentiate both the sides of (4.18) *w.r.t.* y,

$$\frac{\partial}{\partial y} g_{j|r,s}(x,y) G_{s-r}\left(\frac{\overline{F}(y)}{\overline{F}(x)}\Big|\gamma_{r+1},\ldots,\gamma_s\right)$$

$$+ g_{j|r,s}(x,y)\left\{\frac{\partial}{\partial y} G_{s-r}\left(\frac{\overline{F}(y)}{\overline{F}(x)}\Big|\gamma_{r+1},\ldots,\gamma_s\right)\right\}$$

$$= \int_x^y \frac{h(t)}{\overline{F}(t)} G_{j-r}\left(\frac{\overline{F}(t)}{\overline{F}(x)}\Big|\gamma_{r+1},\ldots,\gamma_j\right)$$

$$\times\left\{\frac{\partial}{\partial y} G_{s-j}\left(\frac{\overline{F}(y)}{\overline{F}(t)}\Big|\gamma_{j+1},\ldots,\gamma_s\right)\right\} f(t)dt.$$

Using the relation (A and C), we have

$$\frac{\partial}{\partial y} g_{j|r,s}(x,y) G_{s-r}\left(\frac{\overline{F}(y)}{\overline{F}(x)}\Big|\gamma_{r+1},\ldots,\gamma_s\right)$$

$$- g_{j|r,s}(x,y)\frac{f(y)}{\overline{F}(y)}\left[(\gamma_s - 1) G_{s-r}\left(\frac{\overline{F}(y)}{\overline{F}(x)}\Big|\gamma_{r+1},\ldots,\gamma_s\right)\right.$$

$$\left. - G_{s-r-1}\left(\frac{\overline{F}(y)}{\overline{F}(x)}\Big|\gamma_{r+1},\ldots,\gamma_{s-1}\right)\right]$$

$$= -\frac{f(y)}{\overline{F}(y)} \int_x^y \frac{h(t)}{\overline{F}(t)} G_{j-r}\left(\frac{\overline{F}(t)}{\overline{F}(x)}\Big|\gamma_{r+1},\ldots,\gamma_j\right)$$

$$\times\left[(\gamma_s - 1) G_{s-j}\left(\frac{\overline{F}(y)}{\overline{F}(t)}\Big|\gamma_{j+1},\ldots,\gamma_s\right) - G_{s-j-1}\left(\frac{\overline{F}(y)}{\overline{F}(t)}\Big|\gamma_{j+1},\ldots,\gamma_{s-1}\right)\right] f(t)dt$$

or,

$$\frac{\partial}{\partial y} g_{j|r,s}(x,y) G_{s-r}\left(\frac{\overline{F}(y)}{\overline{F}(x)}\Big|\gamma_{r+1},\ldots,\gamma_s\right)$$

$$-\frac{f(y)}{\overline{F}(y)}\left[g_{j|r,s}(x,y)(\gamma_s - 1) G_{s-r}\left(\frac{\overline{F}(y)}{\overline{F}(x)}\Big|\gamma_{r+1},\ldots,\gamma_s\right)\right.$$

$$\left. - g_{j|r,s}(x,y) G_{s-r-1}\left(\frac{\overline{F}(y)}{\overline{F}(x)}\Big|\gamma_{r+1},\ldots,\gamma_{s-1}\right)\right]$$

$$= -\frac{f(y)}{\overline{F}(y)}\left[g_{j|r,s}(x,y)(\gamma_s - 1)G_{s-r}\left(\frac{\overline{F}(y)}{\overline{F}(x)} | \gamma_{r+1}, \ldots, \gamma_s \right) \right.$$

$$\left. - g_{j|r,s-1}(x,y)G_{s-r-1}\left(\frac{\overline{F}(y)}{\overline{F}(x)} | \gamma_{r+1}, \ldots, \gamma_{s-1} \right) \right]$$

implying that

$$\frac{f(y)}{\overline{F}(y)} \frac{G_{s-r-1}\left(\frac{\overline{F}(y)}{\overline{F}(x)} | \gamma_{r+1}, \ldots, \gamma_{s-1} \right)}{G_{s-r}\left(\frac{\overline{F}(y)}{\overline{F}(x)} | \gamma_{r+1}, \ldots, \gamma_s \right)}$$

$$= \frac{\frac{\partial}{\partial y} g_{j|r,s}(x,y)}{[g_{j|r,s-1}(x,y) - g_{j|r,s}(x,y)]} = D_2(x,y). \tag{4.19}$$

Now using

$$\frac{f(y)}{\overline{F}(y)} G_{s-r-1}\left(\frac{\overline{F}(y)}{\overline{F}(x)} | \gamma_{r+1}, \ldots, \gamma_{s-1} \right) = \frac{\partial}{\partial y} G_{s-r}\left(\frac{\overline{F}(y)}{\overline{F}(x)} | \gamma_{r+1}, \ldots, \gamma_s \right)$$

$$+ \frac{f(y)}{\overline{F}(y)} (\gamma_s - 1) G_{s-r}\left(\frac{\overline{F}(y)}{\overline{F}(x)} | \gamma_{r+1}, \ldots, \gamma_s \right)$$

in (4.19) we get

$$(\gamma_s - 1)\frac{f(y)}{\overline{F}(y)} + \frac{\frac{\partial}{\partial y} G_{s-r}\left(\frac{\overline{F}(y)}{\overline{F}(x)} | \gamma_{r+1}, \ldots, \gamma_s \right)}{G_{s-r}\left(\frac{\overline{F}(y)}{\overline{F}(x)} | \gamma_{r+1}, \ldots, \gamma_s \right)} = D_2(x,y). \tag{4.20}$$

Hence (4.13) is proved.

Integrating both the sides of (4.20) *w.r.t.* y over (y, β) and using the relation (D) we get,

$$(\gamma_s - 1)\log[\overline{F}(t)] + \log G_{s-r}\left(\frac{\overline{F}(t)}{\overline{F}(x)} | \gamma_{r+1}, \ldots, \gamma_s \right) \Bigg|_y^{\beta} = -\int_y^{\beta} D_2(x,t)dt$$

or,

$$\left[\log\left\{[\bar{F}(x)]^{\gamma_s-1}\left[\frac{\bar{F}(t)}{\bar{F}(x)}\right]^{-\gamma_s+1} G_{s-r}\left(\frac{\bar{F}(t)}{\bar{F}(x)}|\gamma_{r+1},\ldots,\gamma_s\right)\right\}\right]_y^\beta$$

$$=-\int_y^\beta D_2(x,t)dt$$

or,

$$\left[\log\left\{[\bar{F}(x)]^{\gamma_s-1} G_{s-r}\left(\frac{\bar{F}(t)}{\bar{F}(x)}|\gamma_{r+1}-\gamma_s+1,\ldots,\gamma_s-\gamma_s+1\right)\right\}\right]_y^\beta$$

$$=-\int_y^\beta D_2(x,t)dt$$

or,

$$\frac{G_{s-r}\left(\frac{\bar{F}(y)}{\bar{F}(x)}|\gamma_{r+1}-\gamma_s+1,\ldots,\gamma_s-\gamma_s+1\right)}{G_{s-r}\left(\frac{\bar{F}(\beta)}{\bar{F}(x)}|\gamma_{r+1}-\gamma_s+1,\ldots,\gamma_s-\gamma_s+1\right)}=\exp\left[-\int_y^\beta D_2(x,t)dt\right]$$

(4.21)

It can be seen that for $\gamma_i > \gamma_s, i = r+1,\ldots,s-1$ (Ahsanullah *et al.*, 2009)

$$G_{s-r}(x\,|\,\gamma_{r+1}-\gamma_s+1,\ldots,\gamma_s-\gamma_s+1) \to a_s^{(r)}(s) \text{ as } x \to 0,$$

and therefore,

$$G_{s-r}\left(\frac{\bar{F}(y)}{\bar{F}(x)}|\gamma_{r+1}-\gamma_s+1,\ldots,\gamma_s-\gamma_s+1\right)=a_s^{(r)}(s)\exp\left[-\int_y^\beta D_2(x,t)dt\right],$$

$$\forall\ \gamma_i > \gamma_s,\ i = r+1,\ldots,s-1. \tag{4.22}$$

For record statistics at $m=-1$, integrating both the sides of (4.20) *w.r.t.* y over (y,q) and noting that for $\gamma_{r+1}=\ldots=\gamma_s$, [Cramer, 2002]:

$$G_{s-r}\left(\frac{\overline{F}(y)}{\overline{F}(x)}\Big|\gamma_{r+1},\ldots,\gamma_s\right)$$

$$=\frac{1}{(s-r-1)!}[-\log\overline{F}(y)+\log\overline{F}(x)]^{s-r-1}\left[\frac{\overline{F}(y)}{\overline{F}(x)}\right]^{\gamma_{r+1}-1},$$

we get

$$\frac{1+\log\overline{F}(y)}{1+\log\overline{F}(x)}=1-\exp\left(-\frac{1}{(s-r-1)}\int_y^q D_2(x,t)\,dt\right),\quad m=-1 \quad (4.23)$$

hence (4.15) is proved.

Corollary 4.2: It may be noted that at $\gamma_i \neq \gamma_j$ but $m_1=\cdots=m_{n-1}=m>-1$

$$\frac{G_{s-r}\left(\frac{\overline{F}(y)}{\overline{F}(x)}\Big|\gamma_{r+1}-\gamma_s+1,\ldots,\gamma_s-\gamma_s+1\right)}{a_s^{(r)}(s)}$$

$$=\frac{\sum_{i=r+1}^{s} a_i^{r}(s)\left(\frac{\overline{F}(y)}{\overline{F}(x)}\right)^{\gamma_i-\gamma_s}}{a_s^{(r)}(s)}$$

$$=\left[1-\frac{\overline{F}(y)^{m+1}}{\overline{F}(x)^{m+1}}\right]^{s-r-1}. \quad (4.24)$$

Proof: Consider, LHS of (4.24)

$$\frac{G_{s-r}\left(\frac{\overline{F}(y)}{\overline{F}(x)}\Big|\gamma_{r+1}-\gamma_s+1,\ldots,\gamma_s-\gamma_s+1\right)}{a_s^{(r)}(s)}$$

$$=\frac{\left[\frac{\overline{F}(y)}{\overline{F}(x)}\right]^{-\gamma_s+1} G_{s-r}\left(\frac{\overline{F}(y)}{\overline{F}(x)}\Big|\gamma_{r+1},\ldots,\gamma_s\right)}{a_s^{(r)}(s)}$$

$$= \frac{\left[\frac{\bar{F}(y)}{\bar{F}(x)}\right]^{-\gamma_s+1} \sum_{i=r+1}^{s} a_i^{(r)}(s) \left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{\gamma_i-1}}{a_s^{(r)}(s)} \quad \text{(Ahsanullah } et\ al.\text{, 2009)}$$

$$= \frac{\sum_{i=r+1}^{s} a_i^{(r)}(s) \left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{\gamma_i-\gamma_s}}{a_s^{(r)}(s)} \tag{4.25}$$

For $m_1 = \ldots = m_{n-1} = m \neq -1$, (4.25) reduces to

$$= \left[1 - \left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{m+1}\right]^{s-r-1}$$

by noting that $a_i^{(r)}(s) = \frac{(-1)^{s-i}}{(m+1)^{s-r-1}(s-r-1)!} \binom{s-r-1}{s-i}$.

Thus (4.14) reduces to

$$\frac{[\bar{F}(y)]^{m+1}}{[\bar{F}(x)]^{m+1}} = 1 - \exp\left(-\frac{1}{(s-r-1)} \int_y^\beta D_2(x,t)dt\right), \quad m > -1 \tag{4.26}$$

as obtained by Ahsanullah *et al.* (2009).

Remark 4.2: With the convention $X(0,n,\tilde{m},k) = x = \alpha$, at $r = 0$, (4.14) reduces to

$$G_s(\bar{F}(y) \mid \gamma_1 - \gamma_s + 1, \ldots, \gamma_s - \gamma_s + 1) = \sum_{i=1}^{s} a_i(s)[\bar{F}(y)]^{\gamma_i - \gamma_s}$$

$$= a_s^{(r)}(s) \exp\left[-\int_y^\beta D(t)dt\right], \ if\ \gamma_{1:s-1>\gamma_s} \tag{4.27}$$

and for $m_1 = \ldots = m_{n-1} = m > -1$

$$[\bar{F}(y)]^{m+1} = 1 - \exp\left[-\frac{1}{(s-1)} \int_y^\beta D(t)dt\right] \tag{4.28}$$

whereas for $\gamma_{r+1} = \ldots = \gamma_s$, [Cramer, 2002]:

$$-\log[\overline{F}(y)] = \exp\left[-\frac{1}{(s-1)}\int_y^q D(t)dt\right], \tag{4.29}$$

where q is as defined in (4.16) and

$$g_{j|s}(x) = E[\{X(j,n,\widetilde{m},k)\} \mid X(s,n,\widetilde{m},k) = y]$$

and

$$D(y) = \frac{g'_{j|s}(y)}{[g_{j|s-1}(y) - g_{j|s}(y)]} = D_2(\alpha, y)$$

as obtained by Bieniek (2009) and Khan and Khan (2011).

For order statistics, Theorem 4.2 reduces as

$$\frac{1-F(x)}{1-F(y)} = 1 - \exp\left[-\frac{1}{(s-r-1)}\int_y^\beta D_2(x,t)dt\right],$$

where $g_{j|r,s}(x,y) = E[h(X_{j:n}) \mid X_{r:n} = x, X_{s:n} = y]$ and $D_2(x,t)$ is as defined in (4.17), as obtained by Khan *et al.* (2007).

At $r = 0, X_{0:n} = x = \alpha$ by convention and hence $F(x) = 0$. We have

$$F(y) = \exp\left(-\frac{1}{(s-1)}\int_y^\beta D(t)dt\right),$$

where

$$g_{j|s}(y) = E[h(X_{j:n}) \mid X_{s:n} = y]$$

and $$D(y) = \frac{g'_{j|s}(y)}{[g_{j|s-1}(y) - g_{j|s}(y)]}$$

as obtained by Khan *et al.* (2006).

Corollary 4.3: Under the assumptions given in Corollary 4.1 and Corollary 4.2,

$$\overline{F}(x)=\left[\frac{e^{I_2}}{e^{I_1}+e^{I_2}-1}\right]^{\frac{1}{m+1}}, \quad m>-1 \tag{4.30}$$

and

$$\overline{F}(y)=\left[\frac{e^{I_2}-1}{e^{I_1}+e^{I_2}-1}\right]^{\frac{1}{m+1}}, \quad m>-1, \tag{4.31}$$

where

$$I_1=\int_\alpha^x A_1(t,y)dt\,,\; I_2=\int_y^\beta A_2(x,t)\,dt$$

and

$$A_1(x,y)=\frac{D_1(x,y)}{(s-r-1)}, \qquad A_2(x,y)=\frac{D_2(x,y)}{(s-r-1)},$$

$$D_1(x,y)=\frac{\frac{\partial}{\partial x}g_{j|r,s}(x,y)}{[g_{j|r,s}(x,y)-g_{j|r+1,s}(x,y)]},$$

$$D_2(x,y)=\frac{\frac{\partial}{\partial y}g_{j|r,s}(x,y)}{[g_{j|r,s-1}(x,y)-g_{j|r,s}(x,y)]}.$$

Proof: From (4.10 and 4.26), we have

$$\frac{1-[\overline{F}(x)]^{m+1}}{1-[\overline{F}(y)]^{m+1}}=1-e^{-I_1} \quad \text{and} \quad \frac{[\overline{F}(y)]^{m+1}}{[\overline{F}(x)]^{m+1}}=1-e^{-I_2}.$$

Therefore,

$$e^{I_1}=\frac{1-[\overline{F}(y)]^{m+1}}{[\overline{F}(x)]^{m+1}-[\overline{F}(y)]^{m+1}}, \quad e^{I_2}=\frac{[\overline{F}(x)]^{m+1}}{[\overline{F}(x)]^{m+1}-[\overline{F}(y)]^{m+1}}$$

and $e^{I_1}+e^{I_2}-1=\dfrac{1}{[\overline{F}(x)]^{m+1}-[\overline{F}(y)]^{m+1}}$

and hence the result.

Corollary 4.4: Under the assumptions given in Corollary 4.1, Corollary 4.2 and Corollary 4.3

$$\overline{F}(x) = \exp\left[-\frac{e^{I_1}-1}{e^{I_1}+e^{I_2}-1}\right], \quad m=-1 \tag{4.32}$$

and

$$\overline{F}(y) = \exp\left[-\frac{e^{I_1}}{e^{I_1}+e^{I_2}-1}\right], \quad m=-1, \tag{4.33}$$

where

$$I_1 = \int_{\alpha}^{x} A_1(t,y)dt, \quad I_2 = \int_{y}^{q} A_2(x,t)\,dt.$$

Proof: The Corollary is proved in view of (4.11) and (4.23).

CHAPTER-IV

CHARACTERIZATION OF DISTRIBUTIONS BY CONDITIONAL EXPECTATION OF DUAL GENERALIZED ORDER STATISTICS

1. Introduction

Distributions have been characterized through the conditional expectations of lower record statistics and dual generalized order statistics by Ahsanullah (2004), Mbah and Ahsanullah (2007), Malinowska and Szynal (2008), Khan *et al.* (2009b, 2010c, 2012c), Shawky and Bakoban (2009) and Khan and Zia (2013) amongst others.

Khan *et al.* (2010a) have characterized the general class of distributions through conditional expectation of dual generalized order statistics conditioned on non-adjacent dual generalized order statistic. We, in this chapter, have extended the result of Khan *et al.*(2010a) for the difference of the conditional expectations conditioned on non-adjacent dual generalized order statistic and for lower record statistic. Further, some of its important deductions are also discussed.

In this chapter, four general class of distributions:

(i) $F(x)=e^{-ah(x)},\ \ x\in(\alpha,\beta);\ \ a>0$

(ii) $F(x)=[1-(m+1)x^{-\alpha}]^{\frac{1}{m+1}},\ \ \beta\le x<\infty;\ \ \alpha>0,$

where $\beta=(m+1)^{\frac{1}{\alpha}}$

(iii) $F(x)=[1-(m+1)e^{-\alpha x}]^{\frac{1}{m+1}},\ \ \beta\le x<\infty;\ \ \alpha>0,$

Parts of the results of this chapter have appeared in Faizan and Khan (2011a, b) and Khan and Faizan (2012).

where $\beta = \frac{1}{\alpha}\ln(m+1)$

(iv) $F(x) = [a\,h(x)+b]^c, \quad x \in (\alpha, \beta)$

have been considered.

2. Characterization theorems based on lower record statistics

Let $X_{L(r)}$ be the r^{th} lower record statistic from a distribution whose *df* is $F(x)$ and *pdf* is $f(x)$, then the conditional *pdf* of $X_{L(s)}$ given $X_{L(r)} = x, \ 1 \le r < s$, is

$$f[X_{L(s)} \mid X_{L(r)} = x] = \frac{1}{\Gamma(s-r)}[-\ln F(y) + \ln F(x)]^{s-r-1}\frac{f(y)}{F(x)},$$

$$x > y. \tag{2.1}$$

Theorem 2.1: Let X be an absolutely continuous random variable with the *df* $F(x)$ and the *pdf* $f(x)$ on the support (α, β), where α and β may be finite or infinite, then for $1 \le r < s$,

$$E[h\{X_{L(s)}\} \mid X_{L(l)} = x] = h(x) + \frac{(s-l)}{a}, \quad l = r, r+1 \tag{2.2}$$

if and only if

$$F(x) = e^{-\,ah(x)}, \quad x \in (\alpha, \beta); \quad a > 0, \tag{2.3}$$

where $h(x)$ is a non-increasing and differentiable function of x with $h(\alpha) = \infty, \ h(\beta) = 0$, such that $h(x)F(x) \to 0$, as $x \to \alpha$.

Proof: For the necessary part, we have

$$E[h\{X_{L(s)}\} \mid X_{L(r)} = x] = \frac{1}{\Gamma(s-r)}\int_{\alpha}^{x} h(y)[-\ln F(y) + \ln F(x)]^{s-r-1}$$

$$\times \frac{f(y)}{F(x)}dy \tag{2.4}$$

$$= -\frac{a}{\Gamma(s-r)} \int_{\alpha}^{x} h(y)[ah(y) - ah(x)]^{s-r-1} h'(y) e^{-a[h(y)-h(x)]} dy .$$

If we let

$h(y) - h(x) = t$, then

$$E[h\{X_{L(s)}\} \mid X_{L(r)} = x] = \frac{a^{s-r}}{\Gamma(s-r)} \int_{0}^{\infty} [t + h(x)] t^{s-r-1} e^{-at} dt .$$

That is,

$$E[h\{X_{L(s)}\} \mid X_{L(r)} = x] = h(x) + \frac{(s-r)}{a} .$$

This proves the necessary part.

For the sufficiency part, we have

$$g_{s|r}(x) = E[h\{X_{L(s)}\} \mid X_{L(r)} = x] = h(x) + \frac{(s-r)}{a} .$$

Hence, in view of (2.1), we have

$$\frac{1}{\Gamma(s-r)} \int_{\alpha}^{x} h(y)[-\ln F(y) + \ln F(x)]^{s-r-1} f(y) dy$$

$$= g_{s|r}(x) F(x) . \tag{2.5}$$

Differentiating both the sides of (2.5) *w.r.t.* x, we get

$$\frac{(s-r-1)}{\Gamma(s-r)} \int_{\alpha}^{x} h(y)[-\ln F(y) + \ln F(x)]^{s-r-2} \frac{f(x)}{F(x)} f(y) dy$$

$$= g'_{s|r}(x) F(x) + g_{s|r}(x) f(x)$$

and hence

$$g_{s|r+1}(x) = g'_{s|r}(x) \frac{F(x)}{f(x)} + g_{s|r}(x) .$$

Therefore,

$$\frac{f(x)}{F(x)} = \frac{g'_{s|r}(x)}{[g_{s|r+1}(x) - g_{s|r}(x)]} \tag{2.6}$$

and

$$F(x) = e^{-\int_x^\beta A(t)dt},$$

where

$$A(t) = \frac{g'_{s|r}(t)}{[g_{s|r+1}(t) - g_{s|r}(t)]} = -ah'(t).$$

Thus,

$$F(x) = e^{-ah(x)}, \quad x \in (\alpha, \beta); \quad a > 0$$

and hence the Theorem.

Theorem 2.2: Under the conditions as given in Theorem 2.1 and for $1 \le r < s < t$,

$$E[h\{X_{L(t)}\} - h\{X_{L(s)}\} \mid X_{L(r)} = x] = \frac{(t-s)}{a} \tag{2.7}$$

if and only if

$$F(x) = e^{-ah(x)}, \quad x \in (\alpha, \beta); \quad a > 0 \tag{2.8}$$

Proof: In view of Theorem 2.1, the necessary part is proved.

For the sufficiency part, let $c^* = \frac{(t-s)}{a}$, then

$$\frac{1}{\Gamma(t-r)} \int_\alpha^x h(y)[-\ln F(y) + \ln F(x)]^{t-r-1} f(y)dy$$

$$- \frac{1}{\Gamma(s-r)} \int_\alpha^x h(y)[-\ln F(y) + \ln F(x)]^{s-r-1} f(y)dy = c^* F(x)$$

(2.9)

Differentiating $(s-r)$ times both the sides of (2.9) *w.r.t.* x, we get

$$\frac{1}{\Gamma(t-s)}\int_{\alpha}^{x} h(y)[-\ln F(y)+\ln F(x)]^{t-s-1}\frac{f(y)}{F(x)}dy = h(x)+c^{*}. \quad (2.10)$$

Integrating left-hand side of (2.10) by parts and simplifying, we have

$$\frac{1}{\Gamma(t-s-1)F(x)}\int_{\alpha}^{x} h(y)[-\ln F(y)+\ln F(x)]^{t-s-2} f(y)dy$$

$$-\frac{1}{\Gamma(t-s)F(x)}\int_{\alpha}^{x} h'(y)[-\ln F(y)+\ln F(x)]^{t-s-1} F(y)dy$$

$$= h(x)+c^{*}. \quad (2.11)$$

This in view of (2.10), reduces to

$$-\frac{a}{\Gamma(t-s)}\int_{\alpha}^{x} h'(y)[-\ln F(y)+\ln F(x)]^{t-s-1} F(y)dy$$

$$= F(x). \quad (2.12)$$

Differentiating (2.12) again $(t-s)$ times *w.r.t.* x, we get

$$\frac{f(x)}{F(x)} = -a\,h'(x)$$

and hence the Theorem.

Remark 2.1: At $s=r$ Theorem 2.2 reduces to Theorem 2.1.

Corollary 2.1: Under the conditions as given in Theorem 2.1 and for $1 \le s < t$,

$$E[h\{X_{L(t)}\}-h\{X_{L(s)}\}]+h(x) = E[h\{X_{L(t)}\} \mid X_{L(s)} = x] \quad (2.13)$$

if and only if

$$F(x) = e^{-ah(x)}, \quad x \in (\alpha,\beta);\ a>0. \quad (2.14)$$

Proof: Corollary follows from Theorem 2.1.

Table 4.1: Examples based on the distribution function $F(x) = e^{-a\,h(x)}$; $a > 0$

Distribution	$F(x)$	a	$h(x)$
Inverse Weibull	$e^{-\theta x^{-p}}$ $0 < x < \infty$	θ	x^{-p}
Power function	$\left(\dfrac{x}{a}\right)^p$ $0 < x < a$	p	$-\log(x/a)$
Logistic	$(1+e^{-x})^{-1}$ $-\infty < x < \infty$	1	$\log(1+e^{-x})$
Burr Type II	$(1+e^{-x})^{-k}$ $-\infty < x < \infty$	k	$\log(1+e^{-x})$
Burr Type III	$(1+x^{-c})^{-k}$ $0 < x < \infty$	k	$\log(1+x^{-c})$
Burr Type IV	$\left[1+\left(\dfrac{c-x}{x}\right)^{1/c}\right]^{-k}$ $0 < x < c$	k	$\log\left[1+\left(\dfrac{c-x}{x}\right)^{1/c}\right]$
Burr Type V	$(1+ce^{-\tan x})^{-k}$ $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$	k	$\log(1+ce^{-\tan x})$
Burr Type VI	$(1+ce^{-k\sinh x})^{-k}$ $-\infty < x < \infty$	k	$\log(1+ce^{-k\sinh x})$
Burr Type VII	$\left(\dfrac{1+\tanh x}{2}\right)^k$ $-\infty < x < \infty$	k	$-\log\left(\dfrac{1+\tanh x}{2}\right)$

Burr Type VIII	$\left(\frac{2}{\pi}\tan^{-1}e^x\right)^k$ $-\infty < x < \infty$	k	$-\log\left(\frac{2}{\pi}\tan^{-1}e^x\right)$
Burr Type X	$(1-e^{-x^2})^k$ $0 < x < \infty$	k	$-\log(1-e^{-x^2})$
Burr Type XI	$(x-\frac{1}{2\pi}\sin 2\pi x)^k$ $0 < x < 1$	k	$-\log(x-\frac{1}{2\pi}\sin 2\pi x)$
Gumbel	$\exp[-e^{-x}]$ $-\infty < x < \infty$	1	e^{-x}
Extreme value II	$e^{-\left(\frac{\theta}{x}\right)^p}$ $0 < x < \infty$	θ^p	x^{-p}

Theorem 2.3: Let X be an absolutely continuous random variable with the *df* $F(x)$ and the *pdf* $f(x)$, then for $1 \le r < s$,

$$E[X_{L(s)}^{-\alpha} \mid X_{L(l)} = x] = a_{s|l}^{*} x^{-\alpha} + b_{s|l}^{*}, \quad l = r, r+1 \tag{2.15}$$

if and only if

$$F(x) = [1-(m+1)x^{-\alpha}]^{\frac{1}{m+1}}, \quad \beta \le x < \infty; \quad \alpha > 0, \tag{2.16}$$

where

$$\beta = (m+1)^{\frac{1}{\alpha}}, \quad a_{s|r}^{*} = \frac{1}{(m+2)^{s-r}} \quad \text{and} \quad b_{s|r}^{*} = \frac{1}{m+1}[1-a_{s|r}^{*}].$$

Proof: We have,

$$E[X_{L(s)}^{-\alpha} \mid X_{L(r)} = x] = \frac{1}{\Gamma(s-r)} \int_{\beta}^{x} y^{-\alpha} [-\ln F(y) + \ln F(x)]^{s-r-1}$$

$$\times \frac{f(y)}{F(x)} dy \tag{2.17}$$

$$= \frac{1}{\Gamma(s-r)} \int_{\beta}^{x} y^{-\alpha} \left[\ln \left[\frac{[1-(m+1)x^{-\alpha}]}{[1-(m+1)y^{-\alpha}]} \right]^{\frac{1}{m+1}} \right]^{s-r-1}$$

$$\times \left[\frac{[1-(m+1)y^{-\alpha}]}{[1-(m+1)x^{-\alpha}]} \right]^{\frac{1}{m+1}} \frac{\alpha\, y^{-(\alpha+1)}}{[1-(m+1)y^{-\alpha}]} dy.$$

Let

$$t = \ln \left[\frac{[1-(m+1)x^{-\alpha}]}{[1-(m+1)y^{-\alpha}]} \right]^{\frac{1}{m+1}}, \text{ then}$$

$$dt = \frac{\alpha\, y^{-(\alpha+1)}}{[1-(m+1)y^{-\alpha}]} dy.$$

Therefore,

$$E[X_{L(s)}^{-\alpha} \mid X_{L(r)} = x] = \frac{1}{\Gamma(s-r)(m+1)}$$

$$\times \int_{0}^{\infty} [1 - \{1-(m+1)x^{-\alpha}\} e^{-(m+1)t}] t^{s-r-1} e^{-t} dt$$

$$= \frac{1}{\Gamma(s-r)(m+1)} \int_{0}^{\infty} t^{s-r-1} e^{-t} dt - \frac{1}{\Gamma(s-r)(m+1)}$$

$$\times [1-(m+1)x^{-\alpha}] \int_{0}^{\infty} t^{s-r-1} e^{-(m+2)t} dt$$

$$= \frac{1}{(m+1)} - \left[\frac{1}{(m+1)} - x^{-\alpha} \right] \frac{1}{(m+2)^{s-r}}$$

$$= \frac{1}{(m+2)^{s-r}} x^{-\alpha} + \frac{1}{(m+1)} \left[1 - \frac{1}{(m+2)^{s-r}} \right]$$

$$E[X_{L(s)}^{-\alpha} \mid X_{L(r)} = x] = a_{s|r}^{*} x^{-\alpha} + b_{s|r}^{*}.$$

This proves the necessary part.

To prove the sufficiency part, we have in view of (2.6)

$$E[X_{L(s)}^{-\alpha} \mid X_{L(r)} = x] = a_{s|r}^{*} x^{-\alpha} + b_{s|r}^{*} = g_{s|r}(x)$$

$$\frac{f(x)}{F(x)} = \frac{g'_{s|r}(x)}{[g_{s|r+1}(x) - g_{s|r}(x)]} = \frac{\alpha x^{-(\alpha+1)}}{[1-(m+1)x^{-\alpha}]}$$

which implies

$$F(x) = [1-(m+1)x^{-\alpha}]^{\frac{1}{m+1}}, \quad \beta \le x < \infty; \quad \alpha > 0$$

and hence the Theorem.

Theorem 2.4: Let X be an absolutely continuous random variable with the *df* $F(x)$ and the *pdf* $f(x)$,then for $1 \le r < s$,

$$E[e^{-\alpha X_{L(s)}} \mid X_{L(l)} = x] = a_{s|l}^{*} e^{-\alpha x} + b_{s|l}^{*}, \quad l = r, r+1 \tag{2.18}$$

if and only if

$$F(x) = [1-(m+1)e^{-\alpha x}]^{\frac{1}{m+1}}, \quad \beta \le x < \infty; \quad \alpha > 0, \tag{2.19}$$

where

$$\beta = \frac{1}{\alpha}\ln(m+1)\,, \quad a_{s|r}^{*} = \frac{1}{(m+2)^{s-r}} \text{ and } b_{s|r}^{*} = \frac{1}{m+1}[1 - a_{s|r}^{*}].$$

Proof: We have,

$$E[e^{-\alpha X_{L(s)}} \mid X_{L(r)} = x]$$

$$= \frac{1}{\Gamma(s-r)} \int_{\beta}^{x} e^{-\alpha y} [-\ln F(y) + \ln F(x)]^{s-r-1}$$

$$\times \frac{f(y)}{F(x)} dy \tag{2.20}$$

$$= \frac{1}{\Gamma(s-r)} \int_{\beta}^{x} e^{-\alpha y} \left[\ln \left[\frac{[1-(m+1)e^{-\alpha x}]}{[1-(m+1)e^{-\alpha y}]} \right]^{\frac{1}{m+1}} \right]^{s-r-1}$$

$$\times \left[\frac{[1-(m+1)e^{-\alpha y}]}{[1-(m+1)e^{-\alpha x}]} \right]^{\frac{1}{m+1}} \frac{\alpha e^{-\alpha y}}{[1-(m+1)e^{-\alpha y}]} dy.$$

Let

$$t = \ln \left[\frac{[1-(m+1)e^{-\alpha x}]}{[1-(m+1)e^{-\alpha y}]} \right]^{\frac{1}{m+1}}, \text{ then}$$

$$dt = \frac{\alpha e^{-\alpha y}}{[1-(m+1)e^{-\alpha y}]} dy.$$

Therefore,

$$E[e^{-\alpha X_{L(s)}} \mid X_{L(r)} = x]$$

$$= \frac{1}{\Gamma(s-r)(m+1)} \int_{0}^{\infty} [1 - \{1-(m+1)e^{-\alpha x}\} e^{-(m+1)t}]$$

$$\times t^{s-r-1} e^{-t} dt$$

$$= \frac{1}{\Gamma(s-r)(m+1)} \int_{0}^{\infty} t^{s-r-1} e^{-t} dt - \frac{1}{\Gamma(s-r)(m+1)}$$

$$\times [1-(m+1)e^{-\alpha x}] \int_{0}^{\infty} t^{s-r-1} e^{-(m+2)t} dt$$

$$= \frac{1}{(m+1)} - \left[\frac{1}{(m+1)} - e^{-\alpha x} \right] \frac{1}{(m+2)^{s-r}}$$

$$= \frac{1}{(m+2)^{s-r}} e^{-\alpha x} + \frac{1}{(m+1)} \left[1 - \frac{1}{(m+2)^{s-r}} \right]$$

$$E[e^{-\alpha X_{L(s)}} \mid X_{L(r)} = x] = a^{*}_{s|r} e^{-\alpha x} + b^{*}_{s|r}.$$

This proves the necessary part.

To prove the sufficiency part, we have in view of (2.6)

$$E[e^{-\alpha X_{L(s)}} \mid X_{L(r)} = x] = a^{*}_{s|r} e^{-\alpha x} + b^{*}_{s|r} = g_{s|r}(x)$$

$$\frac{f(x)}{F(x)} = \frac{g'_{s|r}(x)}{[g_{s|r+1}(x) - g_{s|r}(x)]} = \frac{\alpha e^{-\alpha x}}{[1-(m+1)e^{-\alpha x}]}$$

which implies

$$F(x) = [1-(m+1)e^{-\alpha x}]^{\frac{1}{m+1}}, \quad \beta \le x < \infty; \ \alpha > 0$$

and hence the Theorem.

3. Characterization of distributions based on dual generalized order statistics

Here we will assume two cases:

Case I: $m_1 = m_2 = \ldots = m_{n-1} = m$

Case II: $\gamma_i \ne \gamma_j, \ i \ne j, \ i, j = 1, \ldots, n-1$

3.1. Characterization of distributions when $m_1 = m_2 = \ldots = m_{n-1} = m$

The conditional *pdf* of $X'(s,n,m,k)$ given $X'(r,n,m,k) = x$, $1 \le r < s \le n$, is

$$f_{s|r}(y \mid x) = \frac{c_{s-1}}{c_{r-1}(s-r-1)!(m+1)^{s-r-1}} \left[1 - \left(\frac{F(y)}{F(x)}\right)^{m+1}\right]^{s-r-1}$$

$$\times \left[\frac{F(y)}{F(x)}\right]^{k+(m+1)(n-s)-1} \frac{f(y)}{F(x)}, \quad x > y \tag{3.1}$$

Theorem 3.1: Let $X'(r,n,m,k)$, $r=1,2,\ldots,n$ be the r^{th}-*dgos* from a continuous population with the *df* $F(x)$ and the *pdf* $f(x)$ in the interval (α,β), where α and β may be finite or infinite, then for $1 \le r < s < t \le n$,

$$E[h\{X'(t,n,m,k)\} - h\{X'(s,n,m,k)\} \mid X'(r,n,m,k) = x]$$

$$= \frac{1}{a} \sum_{j=s+1}^{t} \frac{1}{\gamma_j} \tag{3.2}$$

if and only if

$$F(x) = e^{-a\,h(x)}, \quad x \in (\alpha,\beta);\ a > 0, \tag{3.3}$$

where $h(x)$ is a monotonic, non- increasing and differentiable function of x such that $h(x) \to 0$ as $x \to \beta$ and $h(x)F(x) \to 0$ as $x \to \alpha$.

Proof: For the necessary part, we have for $1 \le r < s \le n$, [Khan *et al.*, 2010 a]:

$$E[h\{X'(s,n,m,k)\} \mid X'(r,n,m,k) = x] = h(x) + \frac{1}{a} \sum_{j=r+1}^{s} \frac{1}{\gamma_j}.$$

Therefore for $1 \le r < s < t \le n$,

$$E[h\{X'(t,n,m,k)\} - h\{X'(s,n,m,k)\} \mid X'(r,n,m,k) = x]$$

$$= \left[h(x) + \frac{1}{a} \sum_{j=r+1}^{t} \frac{1}{\gamma_j} \right] - \left[h(x) + \frac{1}{a} \sum_{j=r+1}^{s} \frac{1}{\gamma_j} \right]$$

$$= \frac{1}{a} \sum_{j=s+1}^{t} \frac{1}{\gamma_j}.$$

This proves the necessary part.

For the sufficiency part, let $b = \frac{1}{a} \sum_{j=s+1}^{t} \frac{1}{\gamma_j}$, then

$$\frac{c_{t-1}}{(t-r-1)!c_{r-1}(m+1)^{t-r-1}}\int_\alpha^x h(y)[F(x)^{m+1}-F(y)^{m+1}]^{t-r-1}$$

$$\times[F(y)]^{\gamma_t-1}f(y)dy-\frac{c_{s-1}}{(s-r-1)!c_{r-1}(m+1)^{s-r-1}}$$

$$\times\int_\alpha^x h(y)[F(x)^{m+1}-F(y)^{m+1}]^{s-r-1}[F(y)]^{\gamma_s-1}f(y)dy$$

$$=b\,[F(x)]^{\gamma_{r+1}}. \tag{3.4}$$

Differentiating $(s-r)$ times both the sides of (3.4) *w.r.t.* x, we get

$$\frac{c_{t-1}}{(t-s-1)!c_{s-1}(m+1)^{t-s-1}}$$

$$\times\int_\alpha^x h(y)\frac{[F(x)^{m+1}-F(y)^{m+1}]^{t-s-1}}{[F(x)]^{\gamma_{s+1}}}[F(y)]^{\gamma_t-1}f(y)dy=h(x)+b$$

or,

$$E[h\{X'(t,n,m,k)\}\mid X'(s,n,m,k)=x]=g_{t|s}(x)=h(x)+b.$$

The result [Khan *et al.*, 2010 a],

$$E[h\{X'(t,n,m,k)\}\mid X'(l,n,m,k)=x]=g_{t|l}(x),\ \ l=s,s+1$$

implies

$$F(x)=e^{-\int_x^\beta A(u)\,du}, \tag{3.5}$$

where

$$A(u)=\frac{1}{\gamma_{s+1}}\frac{g'_{t|s}(u)}{[g_{t|s+1}(u)-g_{t|s}(u)]} \tag{3.6}$$

$$=-ah'(u)$$

and we get,

$$F(x)=e^{-ah(x)},\ \ x\in(\alpha,\beta);\ a>0$$

and hence the Theorem.

Remark 3.1: At $r = s$,

$$E[h\{X'(t,n,m,k)\} \mid X'(s,n,m,k) = x] = h(x) + \frac{1}{a} \sum_{j=s+1}^{t} \frac{1}{\gamma_j}$$

as obtained by Khan *et al.* (2010a).

Remark 3.2: At $m = 0, k = 1$, Theorem 3.1 reduces for order statistics as

$$E[h\{(X_{r:n}) - (X_{s:n})\} \mid X_{t:n} = y] = \frac{1}{a} \sum_{j=r}^{s-1} \frac{1}{j}$$

or

$$E[h\{(X_{s:n}) - (X_{r:n})\} \mid X_{t:n} = y] = -\frac{1}{a} \sum_{j=r}^{s-1} \frac{1}{j}$$

if and only if

$$F(x) = e^{-ah(x)}, \quad x \in (\alpha, \beta); \;\; a > 0$$

for $1 \le r < s < t \le n$, as obtained by Khan *et al.* (2013a).

Remark 3.3: At $m = -1$, $k = 1$, Theorem 3.1 reduces for lower record statistics:

$$E[h\{X_{L(t)}\} - h\{X_{L(s)}\} \mid X_{L(r)} = x] = \frac{(t - s)}{a}$$

if and only if

$$F(x) = e^{-ah(x)}, \quad x \in (\alpha, \beta); \;\; a > 0$$

for $1 \le r < s < t$, as obtained in Theorem 2.2.

Theorem 3.2: Let $X'(r,n,m,k)$, $r = 1,2,\ldots,n$ be the r^{th}-*dgos* from a continuous population with the *df* $F(x)$ and the *pdf* $f(x)$, then for $1 \le r < s \le n$,

$$E[X'^{-\alpha}(s,n,m,k) \mid X'(l,n,m,k)=x] = a^*_{s|l}x^{-\alpha}+b^*_{s|l},$$

$$l=r,r+1 \tag{3.7}$$

if and only if

$$F(x)=[1-(m+1)x^{-\alpha}]^{\frac{1}{m+1}}, \;\; \beta \le x < \infty ; \;\; \alpha > 0, \tag{3.8}$$

where

$$\beta=(m+1)^{\frac{1}{\alpha}}, \;\; a^*_{s|r}=\frac{\gamma_s}{\gamma_r} \text{ and } b^*_{s|r}=\frac{1}{m+1}[1-a^*_{s|r}].$$

Proof: We have,

$$E[X'^{-\alpha}(s,n,m,k) \mid X'(r,n,m,k)=x]=\frac{c_{s-1}}{c_{r-1}(s-r-1)!(m+1)^{s-r-1}}$$

$$\times \int_{\beta}^{x} y^{-\alpha}\left[1-\left(\frac{F(y)}{F(x)}\right)^{m+1}\right]^{s-r-1}\left[\frac{F(y)}{F(x)}\right]^{k+(m+1)(n-s)-1}$$

$$\times \frac{f(y)}{F(x)}dy. \tag{3.9}$$

Setting $u=\left[\frac{F(y)}{F(x)}\right]^{m+1}$ then for the df (3.9), it reduces to

$$E[X'^{-\alpha}(s,n,m,k) \mid X'(r,n,m,k)=x]=\frac{c_{s-1}}{c_{r-1}(s-r-1)!(m+1)^{s-r+1}}$$

$$\times \int_0^1 [1-u\{1-(m+1)x^{-\alpha}\}](1-u)^{s-r-1}[u]^{\frac{k+(m+1)(n-s)}{m+1}-1}du$$

$$=\frac{c_{s-1}}{c_{r-1}(s-r-1)!(m+1)^{s-r+1}}\int_0^1 [u]^{\frac{k+(m+1)(n-s)}{m+1}-1}[1-u]^{s-r-1}du$$

$$-\frac{c_{s-1}}{c_{r-1}(s-r-1)!(m+1)^{s-r+1}}[1-(m+1)x^{-\alpha}]$$

$$\times \int_0^1 [u]^{\frac{k+(m+1)(n-s)}{m+1}+1-1}[1-u]^{s-r-1}du$$

$$=\frac{c_{s-1}}{c_{r-1}(s-r-1)!(m+1)^{s-r+1}}B\left(\frac{\gamma_s}{m+1},s-r\right)$$

$$-\frac{c_{s-1}}{c_{r-1}(s-r-1)!(m+1)^{s-r+1}}[1-(m+1)x^{-\alpha}]B\left(\frac{\gamma_s}{m+1}+1,s-r\right)$$

$$=\frac{1}{(m+1)}-\frac{1}{(m+1)}[1-(m+1)x^{-\alpha}]\frac{\gamma_s}{\gamma_r}$$

$$=\frac{\gamma_s}{\gamma_r}x^{-\alpha}+\frac{1}{(m+1)}\left[1-\frac{\gamma_s}{\gamma_r}\right].$$

Thus,

$$E[X'^{-\alpha}(s,n,m,k)\mid X'(r,n,m,k)=x]=a^*_{s|r}x^{-\alpha}+b^*_{s|r}.$$

This proves the necessary part.

To prove the sufficiency part, we have in view of (3.5) and (3.6)

$$\frac{f(x)}{F(x)}=\frac{\alpha x^{-(\alpha+1)}}{[1-(m+1)x^{-\alpha}]}$$

which implies

$$F(x)=[1-(m+1)x^{-\alpha}]^{\frac{1}{m+1}},\quad \beta\le x<\infty;\ \alpha>0$$

and hence the Theorem.

Remark 3.4: At $m=0, k=1$, Theorem 3.2 reduces for $1\le r<s\le n$, order statistics, as

$$E[X_{r:n}^{-\alpha}\mid X_{s:n}=x]=a^*_{r|s}x^{-\alpha}+b^*_{r|s},$$

where

$$a^*_{r|s}=\frac{r}{s}\text{ and } b^*_{r|s}=\left(\frac{s-r}{s}\right)$$

and *df* $F(x)=[1-(m+1)x^{-\alpha}]^{\frac{1}{m+1}}, \beta \le x < \infty ; \alpha > 0$ reduces to $F(x)=1-x^{-\alpha}, 1<x<\infty ; \alpha>0$, which is Pareto distribution.

Remark 3.5: At $m=-1, k=1$, Theorem 3.2 reduces for lower record statistics:

$$E[X_{L(s)}^{-\alpha} \mid X_{L(r)}=x]=x^{-\alpha} \quad \text{with } a_{s|r}^{*}=1 \text{ and } \; b_{s|r}^{*}=0$$

if and only if

$$F(x)=e^{-x^{-\alpha}}, \quad 0<x<\infty; \;\; \alpha>0.$$

Theorem 3.3: Let $X'(r,n,m,k)$, $r=1,2,\ldots,n$ be the $r^{th}-dgos$ from a continuous population with the *df* $F(x)$ and the *pdf* $f(x)$, then for $1 \le r < s < t \le n$,

$$E[X'^{-\alpha}(t,n,m,k) \mid X'(r,n,m,k)=x]$$

$$=a_{t|s}^{*} E[X'^{-\alpha}(s,n,m,k) \mid X'(r,n,m,k)=x]+b_{t|s}^{*} \tag{3.10}$$

if and only if

$$F(x)=[1-(m+1)x^{-\alpha}]^{\frac{1}{m+1}}, \quad \beta \le x < \infty; \;\; \alpha>0, \tag{3.11}$$

where

$$\beta=(m+1)^{\frac{1}{\alpha}}, \quad a_{t|s}^{*}=\frac{\gamma_t}{\gamma_s} \quad \text{and} \quad b_{t|s}^{*}=\frac{1}{m+1}[1-a_{t|s}^{*}].$$

Proof: For the necessary part, we have (in view of Theorem 3.2)

$$E[X^{-\alpha}(t,n,m,k) \mid X'(r,n,m,k)=x] = a_{t|r}^{*} x^{-\alpha}+b_{t|r}^{*}$$

$$=a_{t|r}^{*}\left(x^{-\alpha}-\frac{1}{(m+1)}\right)+\frac{1}{(m+1)}$$

$$= a^*_{t|s} a^*_{s|r} \left(x^{-\alpha} - \frac{1}{(m+1)} \right) + \frac{1}{(m+1)}$$

$$= a^*_{t|s} \left[a^*_{s|r} \left(x^{-\alpha} - \frac{1}{(m+1)} \right) + \frac{1}{(m+1)} \right] - \frac{1}{(m+1)} a^*_{t|s} + \frac{1}{(m+1)}$$

$$= a^*_{t|s} [a^*_{s|r} x^{-\alpha} + b^*_{s|r}] + b^*_{t|s},$$

where

$$a^*_{t|r} = \frac{\gamma_t}{\gamma_s} \times \frac{\gamma_s}{\gamma_r} = a^*_{t|s} a^*_{s|r} \text{ and } b^*_{t|r} = \frac{1}{m+1}[1 - a^*_{t|r}].$$

That is,

$$E[X'^{-\alpha}(t,n,m,k) \mid X'(r,n,m,k) = x]$$

$$= a^*_{t|s} E[X'^{-\alpha}(s,n,m,k) \mid X'(r,n,m,k) = x] + b^*_{t|s}.$$

This proves the necessary part.

For the sufficiency part, we have

$$\frac{c_{t-1}}{c_{r-1}(t-r-1)!(m+1)^{t-r-1}}$$

$$\times \int_\beta^x y^{-\alpha} [F(x)^{m+1} - F(y)^{m+1}]^{t-r-1} [F(y)]^{\gamma_t - 1} f(y) dy$$

$$= a^*_{t|s} \frac{c_{s-1}}{c_{r-1}(s-r-1)!(m+1)^{s-r-1}}$$

$$\times \int_\beta^x y^{-\alpha} [F(x)^{m+1} - F(y)^{m+1}]^{s-r-1} [F(y)]^{\gamma_s - 1} f(y) dy$$

$$+ b^*_{t|s} [F(x)]^{\gamma_{r+1}}. \tag{3.12}$$

Differentiating $(s-r)$ times both the sides of (3.12) $w.r.t.$ x, we get

$$\frac{c_{t-1}}{c_{s-1}(t-s-1)!(m+1)^{t-s-1}[F(x)]^{\gamma_{s+1}}}$$

$$\times \int_{\beta}^{x} y^{-\alpha}[F(x)^{m+1}-F(y)^{m+1}]^{t-s-1}[F(y)]^{\gamma_t -1} f(y)dy$$

$$= a_{t|s}^{*} x^{-\alpha} + b_{t|s}^{*}$$

or,

$$g_{t|s}(x) = a_{t|s}^{*} x^{-\alpha} + b_{t|s}^{*}.$$

Thus, in view of (3.5) and (3.6)

$$\frac{f(x)}{F(x)} = \frac{1}{\gamma_{s+1}} \frac{g'_{t|s}(x)}{[g_{t|s+1}(x)-g_{t|s}(x)]} = \frac{\alpha x^{-(\alpha+1)}}{[1-(m+1)x^{-\alpha}]}$$

which implies

$$F(x) = [1-(m+1)x^{-\alpha}]^{\frac{1}{m+1}}, \quad \beta \le x < \infty; \ \alpha > 0$$

and hence the Theorem.

Remark 3.6: At $r = s$ Theorem 3.3 reduces to Theorem 3.2.

Theorem 3.4: Under the conditions as given in Theorem 3.2 and for $1 \le r < s \le n$,

$$E[e^{-\alpha X'(s,n,m,k)} \mid X'(l,n,m,k) = x] = a_{s|l}^{*} e^{-\alpha x} + b_{s|l}^{*},$$

$$l = r, r+1 \tag{3.13}$$

if and only if

$$F(x) = [1-(m+1)e^{-\alpha x}]^{\frac{1}{m+1}}, \quad \beta \le x < \infty; \ \alpha > 0, \tag{3.14}$$

where

$$\beta = \frac{1}{\alpha}\ln(m+1), \ a_{s|r}^{*} = \frac{\gamma_s}{\gamma_r} \text{ and } b_{s|r}^{*} = \frac{1}{m+1}[1-a_{s|r}^{*}].$$

Proof: We have,

$$E[e^{-\alpha X'(s,n,m,k)} \mid X'(r,n,m,k)=x]$$

$$=\frac{c_{s-1}}{c_{r-1}(s-r-1)!(m+1)^{s-r-1}}\int_{\beta}^{x} e^{-\alpha y}\left[1-\left(\frac{F(y)}{F(x)}\right)^{m+1}\right]^{s-r-1}$$

$$\times\left[\frac{F(y)}{F(x)}\right]^{k+(m+1)(n-s)-1}\frac{f(y)}{F(x)}dy \quad , \quad x>y. \qquad (3.15)$$

Setting $u=\left[\frac{F(y)}{F(x)}\right]^{m+1}$, then it reduces to

$$=\frac{c_{s-1}}{c_{r-1}(s-r-1)!(m+1)^{s-r+1}}\int_{0}^{1}[1-u\{1-(m+1)e^{-\alpha x}\}]$$

$$\times(1-u)^{s-r-1}[u]^{\frac{k+(m+1)(n-s)}{m+1}-1}du$$

$$=\frac{c_{s-1}}{c_{r-1}(s-r-1)!(m+1)^{s-r+1}}\int_{0}^{1}[u]^{\frac{k+(m+1)(n-s)}{m+1}-1}[1-u]^{s-r-1}du$$

$$-\frac{c_{s-1}}{c_{r-1}(s-r-1)!(m+1)^{s-r+1}}[1-(m+1)e^{-\alpha x}]$$

$$\times\int_{0}^{1}[u]^{\frac{k+(m+1)(n-s)}{m+1}+1-1}[1-u]^{s-r-1}du$$

$$=\frac{c_{s-1}}{c_{r-1}(s-r-1)!(m+1)^{s-r+1}}B\left(\frac{\gamma_s}{m+1},s-r\right)$$

$$-\frac{c_{s-1}}{c_{r-1}(s-r-1)!(m+1)^{s-r+1}}[1-(m+1)e^{-\alpha x}]B\left(\frac{\gamma_s}{m+1}+1,s-r\right)$$

$$=\frac{1}{(m+1)}-\frac{1}{(m+1)}[1-(m+1)e^{-\alpha x}]\frac{\gamma_s}{\gamma_r}$$

$$=\frac{\gamma_s}{\gamma_r}e^{-\alpha x}+\frac{1}{(m+1)}\left[1-\frac{\gamma_s}{\gamma_r}\right].$$

Thus,

$$E[e^{-\alpha X'(s,n,m,k)}\mid X'(r,n,m,k)=x]=a^*_{s|r}e^{-\alpha x}+b^*_{s|r}.$$

This proves the necessary part.

For the sufficiency part, using the result (3.5) and (3.6), we have

$$\frac{f(x)}{F(x)}=\frac{\alpha e^{-\alpha x}}{[1-(m+1)e^{-\alpha x}]}$$

which implies

$$F(x)=[1-(m+1)e^{-\alpha x}]^{\frac{1}{m+1}},\quad \beta\le x<\infty;\ \alpha>0$$

and hence the Theorem.

Remark 3.7: At $m=0$, $k=1$, Theorem 3.4 reduces for order statistics: For $1\le r<s\le n$,

$$E[e^{-\alpha X_{r:n}}\mid X_{s:n}=x]=a^*_{r|s}e^{-\alpha x}+b^*_{r|s}$$

if and only if

$$F(x)=1-e^{-\alpha x},\quad 0<x<\infty;\ \alpha>0,$$

where $a^*_{r|s}=\frac{r}{s}$ and $b^*_{r|s}=\left(\frac{s-r}{s}\right)$.

Remark 3.8: At $m=-1$, $k=1$, Theorem 3.4 reduces for lower record statistics: For $1\le r<s$,

$$E[e^{-\alpha X_{L(s)}}\mid X_{L(r)}=x]=e^{-\alpha x}$$

if and only if

$$F(x)=e^{-e^{-\alpha x}},\quad -\infty<x<\infty;\ \alpha>0.$$

Theorem 3.5: Under the conditions as given in Theorem 3.3 and for $1 \le r < s < t \le n$,

$$E[e^{-\alpha X'(t,n,m,k)} \mid X'(r,n,m,k) = x]$$

$$= a^*_{t|s} E[e^{-\alpha X'(s,n,m,k)} \mid X'(r,n,m,k) = x] + b^*_{t|s} \tag{3.16}$$

if and only if

$$F(x) = [1-(m+1)e^{-\alpha x}]^{\frac{1}{m+1}}, \quad \beta \le x < \infty; \ \alpha > 0, \tag{3.17}$$

where

$$\beta = \frac{1}{\alpha}\ln(m+1), \quad a^*_{t|s} = \frac{\gamma_t}{\gamma_s} \ \text{ and } \ b^*_{t|s} = \frac{1}{m+1}[1 - a^*_{t|s}].$$

Proof: For the necessary part, we have (in view of Theorem 3.4)

$$E[e^{-\alpha X'(t,n,m,k)} \mid X'(r,n,m,k) = x] = a^*_{t|r} e^{-\alpha x} + b^*_{t|r}$$

$$= a^*_{t|r}\left(e^{-\alpha x} - \frac{1}{(m+1)}\right) + \frac{1}{(m+1)}$$

$$= a^*_{t|s} a^*_{s|r}\left(e^{-\alpha x} - \frac{1}{(m+1)}\right) + \frac{1}{(m+1)}$$

$$= a^*_{t|s}\left[a^*_{s|r}\left(e^{-\alpha x} - \frac{1}{(m+1)}\right) + \frac{1}{(m+1)}\right] - \frac{1}{(m+1)} a^*_{t|s} + \frac{1}{(m+1)}$$

$$= a^*_{t|s}[a^*_{s|r} e^{-\alpha x} + b^*_{s|r}] + b^*_{t|s},$$

where

$$a^*_{t|r} = \frac{\gamma_t}{\gamma_s} \times \frac{\gamma_s}{\gamma_r} = a^*_{t|s} a^*_{s|r} \quad \text{and} \quad b^*_{t|r} = \frac{1}{m+1}[1 - a^*_{t|r}].$$

That is,

$$E[e^{-\alpha X'(t,n,m,k)} \mid X'(r,n,m,k)=x]$$

$$= a^*_{t|s} E[e^{-\alpha X'(s,n,m,k)} \mid X'(r,n,m,k)=x] + b^*_{t|s}.$$

This proves the necessary part.

For the sufficiency part, we have

$$\frac{c_{t-1}}{c_{r-1}(t-r-1)!(m+1)^{t-r-1}}$$

$$\times \int_\beta^x e^{-\alpha y}[F(x)^{m+1} - F(y)^{m+1}]^{t-r-1}[F(y)]^{\gamma_t - 1} f(y)dy$$

$$= a^*_{t|s} \frac{c_{s-1}}{c_{r-1}(s-r-1)!(m+1)^{s-r-1}}$$

$$\times \int_\beta^x e^{-\alpha y}[F(x)^{m+1} - F(y)^{m+1}]^{s-r-1}[F(y)]^{\gamma_s - 1} f(y)dy$$

$$+ b^*_{t|s}[F(x)]^{\gamma_{r+1}}. \tag{3.18}$$

Differentiating $(s-r)$ times both the sides of (3.18) *w.r.t.* x, we get

$$\frac{c_{t-1}}{c_{s-1}(t-s-1)!(m+1)^{t-s-1}[F(x)]^{\gamma_{s+1}}}$$

$$\times \int_\beta^x e^{-\alpha y}[F(x)^{m+1} - F(y)^{m+1}]^{t-s-1}[F(y)]^{\gamma_t - 1} f(y)dy$$

$$= a^*_{t|s} e^{-\alpha x} + b^*_{t|s}$$

or,

$$g_{t|s}(x) = a^*_{t|s} e^{-\alpha x} + b^*_{t|s}.$$

Thus, in view of (3.5) and (3.6)

$$\frac{f(x)}{F(x)} = \frac{\alpha e^{-\alpha x}}{[1-(m+1)e^{-\alpha x}]}$$

implying that

$$F(x) = [1-(m+1)e^{-\alpha x}]^{\frac{1}{m+1}}, \quad \beta \le x < \infty; \ \alpha > 0$$

and hence the Theorem.

Remark 3.9: At $r = s$ Theorem 3.5 reduces to Theorem 3.4.

Theorem 3.6: Let $X'(r,n,m,k)$, $r = 1,2,\ldots,n$ be the r^{th} *dgos* from a continuous population with the *df* $F(x)$ and the *pdf* $f(x)$ in the interval (α, β), where α and β may be finite or infinite, then for $1 \le r < s < t \le n$,

$$E[h\{X'(t,n,m,k)\} \mid X'(r,n,m,k) = x]$$

$$= a^*_{t|s} E[h\{X'(s,n,m,k)\} \mid X'(r,n,m,k) = x] + b^*_{t|s} \tag{3.19}$$

if and only if

$$F(x) = [ah(x) + b]^c, \quad x \in (\alpha, \beta), \tag{3.20}$$

where

$$a^*_{t|s} = \prod_{j=s+1}^{t} \frac{c\gamma_j}{1 + c\gamma_j} \ \text{ and } \ b^*_{t|s} = -\frac{b}{a}(1 - a^*_{t|s})$$

$h(x)$ is a monotonic and differential function and a, b, c are chosen so that (3.20) is a *df*.

Proof: First we shall prove that (3.20) implies (3.19). In view of Khan *et al.* (2010 a), we have

$$g_{t|r}(x) = E[h\{X'(t,n,m,k)\} \mid X'(r,n,m,k) = x] = a^*_{t|r} h(x) + b^*_{t|r}$$

$$= a^*_{t|r}\left(h(x) + \frac{b}{a}\right) - \frac{b}{a} \tag{3.21}$$

for $F(x)=[ah(x)+b]^c$, $x\in(\alpha,\beta)$,

where

$$a_{t|r}^{*}=\prod_{j=r+1}^{t}\frac{c\gamma_j}{1+c\gamma_j}=a_{t|s}^{*}a_{s|r}^{*} \text{ and } b_{t|r}^{*}=-\frac{b}{a}(1-a_{t|r}^{*}).$$

Therefore,

$$E[h\{X'(t,n,m,k)\}\mid X'(r,n,m,k)=x]=a_{t|s}^{*}a_{s|r}^{*}\left(h(x)+\frac{b}{a}\right)-\frac{b}{a}$$

$$=a_{t|s}^{*}\left[a_{s|r}^{*}\left(h(x)+\frac{b}{a}\right)-\frac{b}{a}\right]-\frac{b}{a}+\frac{b}{a}a_{t|s}^{*}$$

$$=a_{t|s}^{*}E[h\{X'(s,n,m,k)\}\mid X'(r,n,m,k)=x]+b_{t|s}^{*}.$$

This proves the necessary part.

For the sufficiency part, we have

$$\frac{c_{t-1}}{(t-r-1)!c_{r-1}(m+1)^{t-r-1}}\int_{\alpha}^{x}h(y)[F(x)^{m+1}-F(y)^{m+1}]^{t-r-1}$$

$$\times[F(y)]^{\gamma_t-1}f(y)dy=a_{t|s}^{*}\frac{c_{s-1}}{(s-r-1)!c_{r-1}(m+1)^{s-r-1}}$$

$$\times\int_{\alpha}^{x}h(y)[F(x)^{m+1}-F(y)^{m+1}]^{s-r-1}[F(y)]^{\gamma_s-1}f(y)dy$$

$$+b_{t|s}^{*}[F(x)]^{\gamma_{r+1}}. \tag{3.22}$$

Differentiating $(s-r)$ times both the sides of (3.22) $w.r.t.$ x, we get

$$\frac{c_{t-1}}{(t-s-1)!c_{s-1}(m+1)^{t-s-1}}$$

$$\times\int_{\alpha}^{x}h(y)\frac{[F(x)^{m+1}-F(y)^{m+1}]^{t-s-1}}{[F(x)]^{\gamma_{s+1}}}[F(y)]^{\gamma_t-1}f(y)dy$$

$$= a^*_{t|s} h(x) + b^*_{t|s} \tag{3.23}$$

or,

$$g_{t|s}(x) = a^*_{t|s} h(x) + b^*_{t|s}.$$

Therefore, in view of (3.5 and 3.6), we have

$$F(x) = e^{-\int_x^\beta A(u)du}, \tag{3.24}$$

where

$$A(u) = \frac{1}{\gamma_{s+1}} \frac{g'_{t|s}(u)}{[g_{t|s+1}(u) - g_{t|s}(u)]} = \frac{ach'(u)}{[ach(u)+b]}. \tag{3.25}$$

Thus,

$$F(x) = [ah(x)+b]^c, \ \ x \in (\alpha, \beta)$$

and hence the Theorem.

Remark 3.10: At $r = s$, it reduces to the result as obtained by Khan *et al.* (2010 a).

Remark 3.11: Corresponding results for order statistics is,

$$E[h\{(X_{n-t+1:n})\} \mid X_{n-r+1:n} = x]$$

$$= a^*_{n-t+1|n-s+1} E[h\{(X_{n-s+1:n})\} \mid X_{n-r+1:n} = x] + b^*_{n-t+1|n-s+1}$$

or,

$$E[h\{(X_{r:n})\} \mid X_{t:n} = y] = a^*_{r|s} E[h\{(X_{s:n})\} \mid X_{t:n} = y] + b^*_{r|s},$$

where

$$a^*_{r|s} = \prod_{j=r}^{s-1} \frac{cj}{1+cj} \ \text{ and } \ b^*_{r|s} = -\frac{b}{a}(1 - a^*_{r|s})$$

as obtained by Khan *et al.*(2013a).

Remark 3.12: At $m=-1, k=1$, it reduces to the lower record statistics: For $1 \le r < s < t$,

$$E[h\{X_{L(t)}\} \mid X_{L(r)} = x] = a^{*}_{t|s} E[h\{X_{L(s)}\} \mid X_{L(r)} = x] + b^{*}_{t|s} \quad (3.26)$$

if and only if

$$F(x) = [ah(x)+b]^{c}, \quad x \in (\alpha, \beta), \quad (3.27)$$

where

$$a^{*}_{t|s} = \left(\frac{c}{1+c}\right)^{t-s} \text{ and } b^{*}_{t|s} = -\frac{b}{a}(1-a^{*}_{t|s})$$

as obtained by Faizan and Haque (2013).

Remark 3.13: At $a=-\frac{a}{c}$, $b=1$ and $c \to \infty$, $F(x) = [a\,h(x)+b]^{c}$ $\to e^{-a\,h(x)}$ as obtained in Theorem 3.1.

Table 4.2: Examples based on the distribution function $F(x) = [a\,h(x)+b]^{c}$

Distribution	$F(x)$	a	b	c	$h(x)$
Power function	$a^{-p}x^{p}$ $0 < x \le a$	a^{-q}	0	p/q	$x^{q}, q \ne 0$
Pareto	$1-a^{p}x^{-p}$ $a \le x < \infty$	a^{p} $-a^{p}$	$1-a^{p}$ 1	1 1	$1-x^{-p}$ x^{-p}
Inverse Weibull	$e^{-\theta x^{-p}}$ $0 \le x < \infty$	$-\theta/c$ 1	1 0	$c \to \infty$ θ/q	x^{-p} $e^{-q x^{-p}}$, $q \ne 0$
Burr type III	$(1+x^{-c})^{-k}$ $0 \le x < \infty$	1 1	1 0	$-k$ $-k/q$	x^{-c} $(1+x^{-c})^{q}$ $q \ne 0$

Cauchy	$\frac{1}{2}+\frac{1}{\pi}\tan^{-1}x$ $-\infty<x<\infty$	$\frac{1}{\pi}$	$\frac{1}{2}$	1	$\tan^{-1}x$
Generalized Pareto	$[1-(m+1)x^{-\alpha}]^{\frac{1}{(m+1)}}$ $\beta\le x<\infty$	$-(m+1)$	1	$\frac{1}{m+1}$	$x^{-\alpha}$
Generalized exponential	$[1-(m+1)e^{-\alpha x}]^{\frac{1}{(m+1)}}$ $\beta\le x<\infty$	$-(m+1)$	1	$\frac{1}{m+1}$	$e^{-\alpha x}$

Thus the results given in Theorems 3.1, 3.2, 3.3, 3.4 and 3.5 may be reduced from Theorem 3.6.

4. Characterization of distributions when $\gamma_i \neq \gamma_j$; $i \neq j$, $i,j=1,\ldots,n-1$.

The conditional *pdf* of $X'(s,n,\tilde{m},k)$ given $X'(r,n,\tilde{m},k)=x$, $1\le r<s\le n$, is

$$f_{s|r}(y\,|\,x)=\frac{c_{s-1}}{c_{r-1}}\sum_{i=r+1}^{s} a_i^{(r)}(s)\left(\frac{F(y)}{F(x)}\right)^{\gamma_i-1}\frac{f(y)}{F(x)},\quad x>y \tag{4.1}$$

Theorem 4.1: Let $X'(r,n,\tilde{m},k)$, $r=1,2,\ldots,n$ be the r^{th}-*dgos* from a continuous population with the *df* $F(x)$ and the *pdf* $f(x)$ in the interval (α,β), where α and β may be finite or infinite, then for $1\le r<s<t\le n$,

$$E[h\{X'(t,n,\tilde{m},k)\}-h\{X'(s,n,\tilde{m},k)\}\,|\,X'(r,n,\tilde{m},k)=x]$$

$$=\frac{1}{a}\sum_{j=s+1}^{t}\frac{1}{\gamma_j} \tag{4.2}$$

if and only if

$$F(x)=e^{-a\,h(x)},\quad x\in(\alpha,\beta);\ a>0, \tag{4.3}$$

where $h(x)$ is a monotonic and differentiable function of x such that $h(x) \to 0$ as $x \to \beta$ and $h(x)F(x) \to 0$ as $x \to \alpha$.

Proof: Proceeding as in Theorem 3.1, the necessary part follows:

For the sufficiency part, let $b = \frac{1}{a} \sum_{j=s+1}^{t} \frac{1}{\gamma_j}$

$$E[h\{X'(t,n,\widetilde{m},k)\} - h\{X'(s,n,\widetilde{m},k)\} \mid X'(r,n,\widetilde{m},k) = x] = b$$

$$\frac{c_{t-1}}{c_{r-1}} \sum_{i=r+1}^{t} a_i^{(r)}(t) \int_\alpha^x h(y) \left(\frac{F(y)}{F(x)}\right)^{\gamma_i} \frac{f(y)}{F(y)} dy$$

$$- \frac{c_{s-1}}{c_{r-1}} \sum_{i=r+1}^{t} a_i^{(r)}(s) \int_\alpha^x h(y) \left(\frac{F(y)}{F(x)}\right)^{\gamma_i} \frac{f(y)}{F(y)} dy = b. \tag{4.4}$$

Differentiating both the sides of (4.4) $w.r.t.\ x$, we get

$$\frac{f(x)}{F(x)} \frac{c_{t-1}}{c_{r-1}} \sum_{i=r+1}^{t} a_i^{(r)}(t) h(x) - \frac{f(x)}{F(x)} \frac{c_{t-1}}{c_{r-1}} \sum_{i=r+1}^{t} \gamma_i a_i^{(r)}(t)$$

$$\times \int_\alpha^x \frac{h(y) F(y)^{\gamma_i - 1} f(y)}{[F(x)]^{\gamma_i}} dy - \frac{f(x)}{F(x)} \frac{c_{s-1}}{c_{r-1}} \sum_{i=r+1}^{s} a_i^{(r)}(s) h(x)$$

$$+ \frac{f(x)}{F(x)} \frac{c_{s-1}}{c_{r-1}} \sum_{i=r+1}^{s} \gamma_i a_i^{(r)}(s) \int_\alpha^x \frac{h(y) F(y)^{\gamma_i - 1} f(y)}{[F(x)]^{\gamma_i}} dy = 0. \tag{4.5}$$

After noting that $\sum_{i=r+1}^{s} a_i^{(r)}(s) = 0$ [Khan *et al.*, 2006], $c_r = \gamma_{r+1} c_{r-1}$ and $a_i^{(r+1)}(t) = (\gamma_{r+1} - \gamma_i) a_i^{(r)}(t)$, we get

$$\frac{f(x)}{F(x)} \frac{c_{t-1}}{c_{r-1}} \sum_{i=r+1}^{t} \gamma_i a_i^{(r)}(t) \int_\alpha^x \frac{h(y) F(y)^{\gamma_i - 1} f(y)}{[F(x)]^{\gamma_i}} dy$$

$$- \frac{f(x)}{F(x)} \frac{c_{s-1}}{c_{r-1}} \sum_{i=r+1}^{s} \gamma_i a_i^{(r)}(s) \int_\alpha^x \frac{h(y) F(y)^{\gamma_i - 1} f(y)}{[F(x)]^{\gamma_i}} dy = 0$$

$$\frac{f(x)}{F(x)}\frac{\gamma_{r+1}c_{t-1}}{c_{r-1}}\sum_{i=r+1}^{t}a_i^{(r)}(t)\int_{\alpha}^{x}\frac{h(y)F(y)^{\gamma_i-1}f(y)}{[F(x)]^{\gamma_i}}dy$$

$$-\frac{f(x)}{F(x)}\frac{\gamma_{r+1}c_{t-1}}{c_r}\sum_{i=r+2}^{t}a_i^{(r+1)}(t)\int_{\alpha}^{x}\frac{h(y)F(y)^{\gamma_i-1}f(y)}{[F(x)]^{\gamma_i}}dy$$

$$-\frac{f(x)}{F(x)}\frac{\gamma_{s+1}c_{s-1}}{c_{r-1}}\sum_{i=r+1}^{s}a_i^{(r)}(s)\int_{\alpha}^{x}\frac{h(y)F(y)^{\gamma_i-1}f(y)}{[F(x)]^{\gamma_i}}dy$$

$$+\frac{f(x)}{F(x)}\frac{\gamma_{s+1}c_{s-1}}{c_r}\sum_{i=r+2}^{s}a_i^{(r)}(s)\int_{\alpha}^{x}\frac{h(y)F(y)^{\gamma_i-1}f(y)}{[F(x)]^{\gamma_i}}dy=0.$$

That is,

$$\frac{f(x)}{F(x)}\gamma_{r+1}[g_{t|r}(x)-g_{t|r+1}(x)]-\frac{f(x)}{F(x)}\gamma_{r+1}[g_{s|r}(x)-g_{s|r+1}(x)]=0,$$

where

$$g_{s|r}(x)=E[h\{X'(s,n,\widetilde{m},k)\}\,|\,X'(r,n,\widetilde{m},k)=x]$$

or,

$$g_{t|r}(x)-g_{s|r}(x)=g_{t|r+1}(x)-g_{s|r+1}(x)$$

$$=\cdots=g_{t|s}(x)-g_{s|s}(x)=b. \tag{4.6}$$

Noting that $g_{s|s}(x)=h(x)$, we have

$$g_{t|s}(x)=h(x)+b$$

i.e.

$$E[h\{X'(t,n,\widetilde{m},k)\}\,|\,X'(s,n,\widetilde{m},k)=x]=h(x)+\frac{1}{a}\sum_{j=s+1}^{t}\frac{1}{\gamma_j}. \tag{4.7}$$

Using the result [Khan *et al.*, 2010 a], we have

$$F(x)=e^{-\int_x^{\beta}A(u)du}, \tag{4.8}$$

where

$$A(u)=\frac{1}{\gamma_{s+1}}\frac{g'_{t|s}(u)}{[g_{t|s+1}(u)-g_{t|s}(u)]}=-ah'(u) \tag{4.9}$$

and

$$F(x)=e^{-a\,h(x)}, \quad x\in(\alpha,\beta);\; a>0$$

and hence the Theorem.

Remark 4.1: At $r=s$,

$$E[h\{X'(t,n,\widetilde{m},k)\}\,|\,X'(s,n,\widetilde{m},k)=x]=h(x)+\frac{1}{a}\sum_{j=s+1}^{t}\frac{1}{\gamma_j}$$

as obtained by Khan *et al.* (2010 a).

Corollary 4.1: Under the conditions as given in Theorem 4.1 and for $1\le r<s<t\le n$,

$$\begin{aligned}&E[h\{X'(s,n,\widetilde{m},k)\}-h\{X'(r,n,\widetilde{m},k)\}]+h(x)\\&\qquad=E[h\{X'(s,n,\widetilde{m},k)\}\,|\,X'(r,n,\widetilde{m},k)=x]\end{aligned} \tag{4.10}$$

if and only if

$$F(x)=e^{-a\,h(x)}, \quad x\in(\alpha,\beta);\; a>0 \tag{4.11}$$

Proof: Proof follows simply from Theorem 4.1 and Remark 4.1.

Theorem 4.2: Under the conditions as given in Theorem 3.2 and for $1\le r<s\le n$,

$$E[X'^{-\alpha}(s,n,\widetilde{m},k)\,|\,X'(l,n,\widetilde{m},k)=x]=a^{*}_{s|l}x^{-\alpha}+b^{*}_{s|l}, \quad l=r,r+1 \tag{4.12}$$

if and only if

$$F(x)=[1-(m+1)x^{-\alpha}]^{\frac{1}{m+1}}, \quad \beta\le x<\infty;\; \alpha>0, \tag{4.13}$$

where

$$\beta=(m+1)^{\frac{1}{\alpha}},\quad a^{*}_{s|r}=\prod_{j=r+1}^{s}\frac{\gamma_i}{\gamma_j+m+1}\quad \text{and}\quad b^{*}_{s|r}=\frac{1}{m+1}[1-a^{*}_{s|r}].$$

Proof: We have,

$$E[X'^{-\alpha}(s,n,\tilde{m},k)\,|\,X'(r,n,\tilde{m},k)=x]$$

$$=\frac{c_{s-1}}{c_{r-1}}\sum_{i=r+1}^{s}a_i^{(r)}(s)\int_{\beta}^{x}y^{-\alpha}\left[\frac{F(y)}{F(x)}\right]^{\gamma_i-1}\frac{f(y)}{F(x)}dy. \tag{4.14}$$

Setting $u=\left[\frac{F(y)}{F(x)}\right]^{m+1}$, it reduces to

$$E[X'^{-\alpha}(s,n,\tilde{m},k)\,|\,X'(r,n,\tilde{m},k)=x]$$

$$=\frac{c_{s-1}}{c_{r-1}}\sum_{i=r+1}^{s}a_i^{(r)}(s)\frac{1}{(m+1)^2}\int_0^1[1-u\{1-(m+1)x^{-\alpha}\}][u]^{\frac{\gamma_{i-m-1}}{m+1}}du$$

$$=\frac{c_{s-1}}{c_{r-1}}\sum_{i=r+1}^{s}a_i^{(r)}(s)\frac{1}{(m+1)^2}\int_0^1[u]^{\frac{\gamma_{i-m-1}}{m+1}}du$$

$$-\frac{c_{s-1}}{c_{r-1}}\sum_{i=r+1}^{s}a_i^{(r)}(s)\frac{1}{(m+1)^2}[1-(m+1)x^{-\alpha}]\int_0^1[u]^{\frac{\gamma_{i-m-1}}{m+1}+1}du$$

$$=\frac{c_{s-1}}{c_{r-1}}\sum_{i=r+1}^{s}a_i^{(r)}(s)\frac{1}{(m+1)^2}\left[\frac{m+1}{\gamma_i}-\{1-(m+1)x^{-\alpha}\}\frac{m+1}{\gamma_{i+m+1}}\right]$$

$$=\frac{c_{s-1}}{c_{r-1}}\sum_{i=r+1}^{s}\frac{a_i^{(r)}(s)}{\gamma_{i+m+1}}x^{-\alpha}+\frac{1}{(m+1)}\left[1-\frac{c_{s-1}}{c_{r-1}}\sum_{i=r+1}^{s}\frac{a_i^{(r)}(s)}{\gamma_{i+m+1}}\right]$$

$$=a^{*}_{s|r}x^{-\alpha}+b^{*}_{s|r}.$$

Thus,

$$E[X'^{-\alpha}(s,n,\tilde{m},k)\,|\,X'(r,n,\tilde{m},k)=x]=a^{*}_{s|r}x^{-\alpha}+b^{*}_{s|r}.$$

This proves the necessary part.

To prove the sufficiency part, we have [Khan *et al.*, 2010 a]:

$$\frac{f(x)}{F(x)}=\frac{1}{\gamma_{r+1}}\frac{g'_{s|r}(x)}{[g_{s|r+1}(x)-g_{s|r}(x)]}=\frac{\alpha x^{-(\alpha+1)}}{[1-(m+1)x^{-\alpha}]}$$

which implies

$$F(x)=[1-(m+1)x^{-\alpha}]^{\frac{1}{m+1}}, \quad \beta\le x<\infty;\ \alpha>0$$

and hence the Theorem.

Theorem 4.3: Under the conditions as given in Theorem 3.3 and for $1\le r<s<t\le n$,

$$E[X'^{-\alpha}(t,n,\tilde{m},k)\,|\,X'(r,n,\tilde{m},k)=x]$$

$$=a^{*}_{t|s}E[X'^{-\alpha}(s,n,\tilde{m},k)\,|\,X'(r,n,\tilde{m},k)=x]+b^{*}_{t|s} \tag{4.15}$$

if and only if

$$F(x)=[1-(m+1)x^{-\alpha}]^{\frac{1}{m+1}}, \quad \beta\le x<\infty;\ \alpha>0, \tag{4.16}$$

where

$$\beta=(m+1)^{\frac{1}{\alpha}},\ a^{*}_{t|s}=\prod_{j=s+1}^{t}\frac{\gamma_j}{\gamma_j+m+1} \ \text{ and } \ b^{*}_{t|s}=\frac{1}{m+1}[1-a^{*}_{t|s}].$$

Proof: Necessary part follows from Theorem 3.3.

For the sufficiency part, we have

$$\frac{c_{t-1}}{c_{r-1}}\sum_{i=r+1}^{t}a_i^{(r)}(t)\int_{\beta}^{x}y^{-\alpha}\left[\frac{F(y)}{F(x)}\right]^{\gamma_i}\frac{f(y)}{F(y)}dy$$

$$=a^{*}_{t|s}\frac{c_{s-1}}{c_{r-1}}\sum_{i=r+1}^{s}a_i^{(r)}(s)\int_{\beta}^{x}y^{-\alpha}\left[\frac{F(y)}{F(x)}\right]^{\gamma_i}\frac{f(y)}{F(y)}dy+b^{*}_{t|s}. \tag{4.17}$$

Differentiating both the sides of (4.17) *w.r.t.* x, we get

$$\left[\frac{f(x)}{F(x)}\frac{c_{t-1}}{c_{r-1}}\sum_{i=r+1}^{t}a_i^{(r)}(t)x^{-\alpha}-\frac{f(x)}{F(x)}\frac{c_{t-1}}{c_{r-1}}\sum_{i=r+1}^{t}\gamma_i a_i^{(r)}(t)\right.$$

$$\left.\times\int_\beta^x\frac{y^{-\alpha}[F(y)]^{\gamma_i-1}f(y)}{[F(x)]^{\gamma_i}}dy\right]$$

$$=a_{t|s}^{*}\left[\frac{f(x)}{F(x)}\frac{c_{s-1}}{c_{r-1}}\sum_{i=r+1}^{s}a_i^{(r)}(s)x^{-\alpha}-\frac{f(x)}{F(x)}\frac{c_{s-1}}{c_{r-1}}\sum_{i=r+1}^{s}\gamma_i a_i^{(r)}(s)\right.$$

$$\left.\times\int_\beta^x\frac{y^{-\alpha}[F(y)]^{\gamma_i-1}f(y)}{[F(x)]^{\gamma_i}}dy\right].$$

After noting that $\sum_{i=r+1}^{s}a_i^{(r)}(s)=0$ [Khan *et al.*, 2006], $c_r=\gamma_{r+1}c_{r-1}$ and $a_i^{(r+1)}(t)=(\gamma_{r+1}-\gamma_i)a_i^{(r)}(t)$, we get

$$\frac{f(x)}{F(x)}\frac{\gamma_{r+1}c_{t-1}}{c_{r-1}}\sum_{i=r+1}^{t}a_i^{(r)}(t)\int_\beta^x\frac{y^{-\alpha}[F(y)]^{\gamma_i-1}f(y)}{[F(x)]^{\gamma_i}}dy$$

$$-\frac{f(x)}{F(x)}\frac{\gamma_{r+1}c_{t-1}}{c_r}\sum_{i=r+2}^{t}a_i^{(r+1)}(t)\int_\beta^x\frac{y^{-\alpha}[F(y)]^{\gamma_i-1}f(y)}{[F(x)]^{\gamma_i}}dy$$

$$=a_{t|s}^{*}\left[\frac{f(x)}{F(x)}\frac{\gamma_{r+1}c_{s-1}}{c_{r-1}}\sum_{i=r+1}^{s}a_i^{(r)}(s)\int_\beta^x\frac{y^{-\alpha}[F(y)]^{\gamma_i-1}f(y)}{[F(x)]^{\gamma_i}}dy\right.$$

$$\left.-\frac{f(x)}{F(x)}\frac{\gamma_{r+1}c_{s-1}}{c_r}\sum_{i=r+2}^{s}a_i^{(r+1)}(s)\int_\beta^x\frac{y^{-\alpha}[F(y)]^{\gamma_i-1}f(y)}{[F(x)]^{\gamma_i}}dy\right].$$

That is,

$$\frac{f(x)}{F(x)}\gamma_{r+1}[g_{t|r}(x)-g_{t|r+1}(x)]=\frac{f(x)}{F(x)}\gamma_{r+1}a_{t|s}^{*}[g_{s|r}(x)-g_{s|r+1}(x)],$$

where

$$E[X'^{-\alpha}(s,n,\tilde{m},k) \mid X'(r,n,\tilde{m},k)=x]=g_{s|r}(x)$$

or,

$$g_{t|r}(x)-a^*_{t|s}\, g_{s|r}(x)=g_{t|r+1}(x)-a^*_{t|s}\, g_{s|r+1}(x)$$

$$=\cdots=g_{t|s}(x)-a^*_{t|s}(x)g_{s|s}(x)=b^*_{t|s}.$$

Noting that $g_{s|s}(x)=x^{-\alpha}$, we have

$$g_{t|s}(x)=E[X'^{-\alpha}(t,n,\tilde{m},k) \mid X'(s,n,\tilde{m},k)=x] = a^*_{t|s}x^{-\alpha}+b^*_{t|s}.$$

Using the result [Khan *et al.*, 2010 a], we have

$$\frac{f(x)}{F(x)}=\frac{\alpha x^{-(\alpha+1)}}{[1-(m+1)x^{-\alpha}]}$$

which implies

$$F(x)=[1-(m+1)x^{-\alpha}]^{\frac{1}{m+1}}, \quad \beta \le x<\infty;\ \alpha>0$$

and hence the Theorem.

Theorem 4.4: Under the conditions as given in Theorem 3.2 and for $1 \le r<s \le n$,

$$E[e^{-\alpha X'(s,n,\tilde{m},k)} \mid X'(l,n,\tilde{m},k)=x] = a^*_{s|l}e^{-\alpha x}+b^*_{s|l},$$

$$l=r,r+1 \tag{4.18}$$

if and only if

$$F(x)=[1-(m+1)e^{-\alpha x}]^{\frac{1}{m+1}}, \quad \beta \le x<\infty;\ \alpha>0, \tag{4.19}$$

where

$$\beta=\frac{1}{\alpha}\ln(m+1),\ a^*_{s|r}=\prod_{j=r+1}^{s}\frac{\gamma_j}{\gamma_j+m+1} \text{ and } b^*_{s|r}=\frac{1}{m+1}[1-a^*_{s|r}].$$

Proof: We have,

$$E[e^{-\alpha X'(s,n,\tilde{m},k)} \mid X'(r,n,\tilde{m},k)=x]$$

$$=\frac{c_{s-1}}{c_{r-1}}\sum_{i=r+1}^{s}a_i^{(r)}(s)\int_\beta^x e^{-\alpha y}\left[\frac{F(y)}{F(x)}\right]^{\gamma_i-1}\frac{f(y)}{F(x)}dy. \tag{4.20}$$

Setting $u=\left[\dfrac{F(y)}{F(x)}\right]^{m+1}$, it reduces to

$$E[e^{-\alpha X'(s,n,\tilde{m},k)} \mid X'(r,n,\tilde{m},k)=x]$$

$$=\frac{c_{s-1}}{c_{r-1}}\sum_{i=r+1}^{s}a_i^{(r)}(s)\frac{1}{(m+1)^2}\int_0^1[1-u\{1-(m+1)e^{-\alpha x}\}][u]^{\frac{\gamma_{i-m-1}}{m+1}}du$$

$$=\frac{c_{s-1}}{c_{r-1}}\sum_{i=r+1}^{s}a_i^{(r)}(s)\frac{1}{(m+1)^2}\int_0^1[u]^{\frac{\gamma_{i-m-1}}{m+1}}du$$

$$-\frac{c_{s-1}}{c_{r-1}}\sum_{i=r+1}^{s}a_i^{(r)}(s)\frac{1}{(m+1)^2}[1-(m+1)e^{-\alpha x}]\int_0^1[u]^{\frac{\gamma_{i-m-1}}{m+1}+1}du$$

$$=\frac{c_{s-1}}{c_{r-1}}\sum_{i=r+1}^{s}a_i^{(r)}(s)\frac{1}{(m+1)^2}\left[\frac{m+1}{\gamma_i}-\{1-(m+1)e^{-\alpha x}\}\frac{m+1}{\gamma_{i+m+1}}\right]$$

$$=\frac{c_{s-1}}{c_{r-1}}\sum_{i=r+1}^{s}\frac{a_i^{(r)}(s)}{\gamma_{i+m+1}}e^{-\alpha x}+\frac{1}{(m+1)}\left[1-\frac{c_{s-1}}{c_{r-1}}\sum_{i=r+1}^{s}\frac{a_i^{(r)}(s)}{\gamma_{i+m+1}}\right]$$

$$=a_{s|r}^{*}e^{-\alpha x}+b_{s|r}^{*}.$$

Thus,

$$E[e^{-\alpha X'(s,n,\tilde{m},k)} \mid X'(r,n,\tilde{m},k)=x]=a_{s|r}^{*}e^{-\alpha x}+b_{s|r}^{*}.$$

This proves the necessary part.

For the sufficiency part, we have [Khan *et al.*, 2010 a]:

$$\frac{f(x)}{F(x)}=\frac{1}{\gamma_{r+1}}\frac{g'_{s|r}(x)}{[g_{s|r+1}(x)-g_{s|r}(x)]}=\frac{\alpha e^{-\alpha x}}{[1-(m+1)e^{-\alpha x}]}$$

implying that

$$F(x)=[1-(m+1)e^{-\alpha x}]^{\frac{1}{m+1}},\quad \beta\le x<\infty;\ \alpha>0$$

and hence the Theorem.

Theorem 4.5: Under the conditions as given in Theorem 3.3 and for $1\le r<s<t\le n$,

$$E[e^{-\alpha X'(t,n,\tilde{m},k)}\mid X'(r,n,\tilde{m},k)=x]$$

$$=a^{*}_{t|s}E[e^{-\alpha X'(s,n,\tilde{m},k)}\mid X'(r,n,\tilde{m},k)=x]+b^{*}_{t|s} \tag{4.21}$$

if and only if

$$F(x)=[1-(m+1)e^{-\alpha x}]^{\frac{1}{m+1}},\quad \beta\le x<\infty;\ \alpha>0, \tag{4.22}$$

where

$$\beta=\frac{1}{\alpha}\ln(m+1),\ \ a^{*}_{t|s}=\prod_{j=s+1}^{t}\frac{\gamma_j}{\gamma_j+m+1}\ \text{ and }\ b^{*}_{t|s}=\frac{1}{m+1}[1-a^{*}_{t|s}].$$

Proof: Necessary part follows from Theorem 3.5.

For the sufficiency part, we have

$$\frac{c_{t-1}}{c_{r-1}}\sum_{i=r+1}^{t}a_i^{(r)}(t)\int_{\beta}^{x}e^{-\alpha y}\left[\frac{F(y)}{F(x)}\right]^{\gamma_i}\frac{f(y)}{F(y)}dy$$

$$=a^{*}_{t|s}\frac{c_{s-1}}{c_{r-1}}\sum_{i=r+1}^{s}a_i^{(r)}(s)\int_{\beta}^{x}e^{-\alpha y}\left[\frac{F(y)}{F(x)}\right]^{\gamma_i}\frac{f(y)}{F(y)}dy+b^{*}_{t|s}. \tag{4.23}$$

Differentiating both the sides of (4.23) *w.r.t.* x, we get

$$\left[\frac{f(x)}{F(x)}\frac{c_{t-1}}{c_{r-1}}\sum_{i=r+1}^{t}a_i^{(r)}(t)e^{-\alpha x}-\frac{f(x)}{F(x)}\frac{c_{t-1}}{c_{r-1}}\sum_{i=r+1}^{t}\gamma_i a_i^{(r)}(t)\right.$$

$$\left.\times\int_\beta^x\frac{e^{-\alpha y}[F(y)]^{\gamma_i-1}f(y)}{[F(x)]^{\gamma_i}}\,dy\right]$$

$$=a_{t|s}^{*}\left[\frac{f(x)}{F(x)}\frac{c_{s-1}}{c_{r-1}}\sum_{i=r+1}^{s}a_i^{(r)}(s)e^{-\alpha x}-\frac{f(x)}{F(x)}\frac{c_{s-1}}{c_{r-1}}\sum_{i=r+1}^{s}\gamma_i a_i^{(r)}(s)\right.$$

$$\left.\times\int_\beta^x\frac{e^{-\alpha y}[F(y)]^{\gamma_i-1}f(y)}{[F(x)]^{\gamma_i}}\,dy\right].$$

After noting that $\sum_{i=r+1}^{s}a_i^{(r)}(s)=0$ [Khan *et al.*, 2006], $c_r=\gamma_{r+1}c_{r-1}$ and $a_i^{(r+1)}(t)=(\gamma_{r+1}-\gamma_i)a_i^{(r)}(t)$, we get

$$\frac{f(x)}{F(x)}\frac{\gamma_{r+1}c_{t-1}}{c_{r-1}}\sum_{i=r+1}^{t}a_i^{(r)}(t)\int_\beta^x\frac{e^{-\alpha y}[F(y)]^{\gamma_i-1}f(y)}{[F(x)]^{\gamma_i}}\,dy$$

$$-\frac{f(x)}{F(x)}\frac{\gamma_{r+1}c_{t-1}}{c_r}\sum_{i=r+2}^{t}a_i^{(r+1)}(t)\int_\beta^x\frac{e^{-\alpha y}[F(y)]^{\gamma_i-1}f(y)}{[F(x)]^{\gamma_i}}\,dy$$

$$=a_{t|s}^{*}\left[\frac{f(x)}{F(x)}\frac{\gamma_{r+1}c_{s-1}}{c_{r-1}}\sum_{i=r+1}^{s}a_i^{(r)}(s)\int_\beta^x\frac{e^{-\alpha y}[F(y)]^{\gamma_i-1}f(y)}{[F(x)]^{\gamma_i}}\,dy\right.$$

$$\left.-\frac{f(x)}{F(x)}\frac{\gamma_{r+1}c_{s-1}}{c_r}\sum_{i=r+2}^{s}a_i^{(r+1)}(s)\int_\beta^x\frac{e^{-\alpha y}[F(y)]^{\gamma_i-1}f(y)}{[F(x)]^{\gamma_i}}\,dy\right].$$

That is,

$$\frac{f(x)}{F(x)}\gamma_{r+1}[g_{t|r}(x)-g_{t|r+1}(x)]=\frac{f(x)}{F(x)}\gamma_{r+1}a_{t|s}^{*}[g_{s|r}(x)-g_{s|r+1}(x)],$$

where

$$E[e^{-\alpha X'(s,n,\widetilde{m},k)}\mid X'(r,n,\widetilde{m},k)=x]=g_{s|r}(x)$$

or,

$$g_{t|r}(x) - a^*_{t|s} g_{s|r}(x) = g_{t|r+1}(x) - a^*_{t|s} g_{s|r+1}(x)$$

$$= \cdots = g_{t|s}(x) - a^*_{t|s}(x) g_{s|s}(x) = b^*_{t|s}.$$

Noting that $g_{s|s}(x) = e^{-\alpha x}$, we have

$$E[e^{-\alpha X'(t,n,\tilde{m},k)} \mid X'(s,n,\tilde{m},k) = x] = a^*_{t|s} e^{-\alpha x} + b^*_{t|s}$$

$$g_{t|s}(x) = a^*_{t|s} e^{-\alpha x} + b^*_{t|s}.$$

Using the result [Khan *et al.*, 2010 a], we have

$$\frac{f(x)}{F(x)} = \frac{\alpha e^{-\alpha x}}{[1-(m+1)e^{-\alpha x}]}$$

which implies

$$F(x) = [1-(m+1)e^{-\alpha x}]^{\frac{1}{m+1}}, \quad \beta \le x < \infty; \ \alpha > 0$$

and hence the Theorem.

Theorem 4.6: Let $X'(r,n,\tilde{m},k)$, $r = 1,2,\ldots,n$ be the r^{th}-*dgos* from a continuous population with the *df* $F(x)$ and the *pdf* $f(x)$ in the interval (α,β), where α and β may be finite or infinite, then for $1 \le r < s < t \le n$,

$$E[h\{X'(t,n,\tilde{m},k)\} \mid X'(r,n,\tilde{m},k) = x]$$

$$= a^*_{t|s} E[h\{X'(s,n,\tilde{m},k)\} \mid X'(r,n,\tilde{m},k) = x] + b^*_{t|s} \tag{4.24}$$

if and only if

$$F(x) = [a\,h(x) + b]^c \ , x \in (\alpha,\beta) \tag{4.25}$$

where

$$a_{t|s}^{*} = \prod_{j=s+1}^{t} \frac{c\gamma_j}{1+c\gamma_j} \text{ and } b_{t|s}^{*} = -\frac{b}{a}(1-a_{t|s}^{*})$$

Proof: Proceeding as in the Theorem 3.6, the necessary part follows.

For the sufficiency part, we have

$$\frac{c_{t-1}}{c_{r-1}} \sum_{i=r+1}^{t} a_i^{(r)}(t) \int_{\alpha}^{x} h(y) \left[\frac{F(y)}{F(x)}\right]^{\gamma_i} \frac{f(y)}{F(y)} dy$$

$$= a_{t|s}^{*} \frac{c_{s-1}}{c_{r-1}} \sum_{i=r+1}^{s} a_i^{(r)}(s) \int_{\alpha}^{x} h(y) \left[\frac{F(y)}{F(x)}\right]^{\gamma_i} \frac{f(y)}{F(y)} dy + b_{t|s}^{*}. \tag{4.26}$$

Differentiating both the sides of (4.26) *w.r.t.* x, we get

$$\left[\frac{f(x)}{F(x)} \frac{c_{t-1}}{c_{r-1}} \sum_{i=r+1}^{t} a_i^{(r)}(t) h(x) - \frac{f(x)}{F(x)} \frac{c_{t-1}}{c_{r-1}} \sum_{i=r+1}^{t} \gamma_i a_i^{(r)}(t) \right.$$
$$\left. \times \int_{\alpha}^{x} \frac{h(y)[F(y)]^{\gamma_i - 1} f(y)}{[F(x)]^{\gamma_i}} dy \right]$$

$$= a_{t|s}^{*} \left[\frac{f(x)}{F(x)} \frac{c_{s-1}}{c_{r-1}} \sum_{i=r+1}^{s} a_i^{(r)}(s) h(x) - \frac{f(x)}{F(x)} \frac{c_{s-1}}{c_{r-1}} \sum_{i=r+1}^{s} \gamma_i a_i^{(r)}(s) \right.$$
$$\left. \times \int_{\alpha}^{x} \frac{h(y)[F(y)]^{\gamma_i - 1} f(y)}{[F(x)]^{\gamma_i}} dy \right].$$

After noting that $\sum_{i=r+1}^{s} a_i^{(r)}(s) = 0$ [Khan *et al.*, 2006], $c_r = \gamma_{r+1} c_{r-1}$ and $a_i^{(r+1)}(t) = (\gamma_{r+1} - \gamma_i) a_i^{(r)}(t)$, we get

$$\frac{f(x)}{F(x)} \frac{\gamma_{r+1} c_{t-1}}{c_{r-1}} \sum_{i=r+1}^{t} a_i^{(r)}(t) \int_{\alpha}^{x} \frac{h(y)[F(y)]^{\gamma_i - 1} f(y)}{[F(x)]^{\gamma_i}} dy$$

$$- \frac{f(x)}{F(x)} \frac{\gamma_{r+1} c_{t-1}}{c_r} \sum_{i=r+2}^{t} a_i^{(r+1)}(t) \int_{\alpha}^{x} \frac{h(y)[F(y)]^{\gamma_i - 1} f(y)}{[F(x)]^{\gamma_i}} dy$$

$$= a^*_{t|s}\left[\frac{f(x)}{F(x)}\frac{\gamma_{r+1}c_{s-1}}{c_{r-1}}\sum_{i=r+1}^{s}a_i^{(r)}(s)\int_\alpha^x \frac{h(y)[F(y)]^{\gamma_i-1}f(y)}{[F(x)]^{\gamma_i}}dy\right.$$

$$\left.-\frac{f(x)}{F(x)}\frac{\gamma_{r+1}c_{s-1}}{c_r}\sum_{i=r+2}^{s}a_i^{(r+1)}(s)\int_\alpha^x \frac{h(y)[F(y)]^{\gamma_i-1}f(y)}{[F(x)]^{\gamma_i}}dy\right]$$

That is,

$$\frac{f(x)}{F(x)}\gamma_{r+1}[g_{t|r}(x)-g_{t|r+1}(x)] = \frac{f(x)}{F(x)}\gamma_{r+1}a^*_{t|s}[g_{s|r}(x)-g_{s|r+1}(x)],$$

where

$$g_{s|r}(x) = E[h\{X'(s,n,\tilde{m},k)\} \mid X'(r,n,\tilde{m},k)=x]$$

or,

$$g_{t|r}(x) - a^*_{t|s}\, g_{s|r}(x) = g_{t|r+1}(x) - a^*_{t|s}\, g_{s|r+1}(x)$$

$$= \cdots = g_{t|s}(x) - a^*_{t|s}\, g_{s|s}(x) = b^*_{t|s}. \tag{4.27}$$

Noting that $g_{s|s}(x) = h(x)$, we have

$$g_{t|s}(x) = E[h\{X'(t,n,\tilde{m},k)\} \mid X'(s,n,\tilde{m},k)=x]$$

$$= a^*_{t|s}h(x) + b^*_{t|s}. \tag{4.28}$$

Therefore, in view of (3.5 and 3.6), we have

$$F(x) = e^{-\int_x^\beta A(u)du}, \tag{4.29}$$

where

$$A(u) = \frac{1}{\gamma_{s+1}}\frac{g'_{t|s}(u)}{[g_{t|s+1}(u)-g_{t|s}(u)]} = \frac{ach'(u)}{[ach(u)+b]}. \tag{4.30}$$

Thus,

$$F(x) = [ah(x)+b]^c, \quad x \in (\alpha,\beta)$$

and hence the Theorem.

Remark 4.2: It may be seen that when $\gamma_i \neq \gamma_j$ but $m_i = m_j = m$, then [Khan *et al.*, 2010 a]:

$$a_i^{(r)}(s) = \frac{1}{(m+1)^{s-r-1}(s-r-1)!}(-1)^{s-i}\binom{s-r-1}{s-i}.$$

Therefore (4.1) reduces to

$$f_{s|r}(y|x) = \frac{c_{s-1}}{c_{r-1}(s-r-1)!(m+1)^{s-r-1}}$$

$$\times\left[1-\left(\frac{F(y)}{F(x)}\right)^{m+1}\right]^{s-r-1}\left[\left(\frac{F(y)}{F(x)}\right)\right]^{k+(m+1)(n-s)-1}\frac{f(y)}{F(x)}$$

as given in Chapter-III, which is essentially (3.1).That is conditional *pdf* given in (4.1) reduces to the conditional *pdf* given in (3.1). Thus the results of Section 4 may be reduced to the results of Section 3.1.

CHAPTER-V

EXACT MOMENTS OF RECORD VALUES FROM BURR DISTRIBUTION WITH APPLICATIONS

1. Introduction

Khan and Khan (1987) have obtained the recurrence relations for single and product moments of order statistics from the Burr distribution. Khan and Ali (1995) extended the results of Khan and Khan (1987) to include the case of negative and ratio moments of order statistics. Pawlas and Szynal (1999), Saran and Pushkarna (2000) obtained the recurrence relations for single and product moments of k^{th} record values from Burr distribution.

In this chapter, we have obtained exact single and product moments for record statistics from Burr distribution and have tabulated means, variances and covariances for some selected records using MATLAB and have used them to obtain BLUE and BLUP for parameters of Burr distribution. For application of the Burr distribution, one may refer to Tadikamalla (1980).

A random variable X is said to have the Burr distribution if the *pdf* of X is of the form

$$f(x) = \mu p x^{p-1}[1+x^p]^{-(\mu+1)}, \quad x>0; \ \mu, p>0 \qquad (1.1)$$

$$= 0, \qquad \text{otherwise}$$

with the corresponding *df*

$$F(x) = 1-[1+x^p]^{-\mu}. \qquad (1.2)$$

Therefore, for the Burr distribution, we have

Part of the results of this chapter are contained in Khan *et al.* (2013c).

$$f(x) = \frac{\mu p x^{p-1}}{[1+x^p]} \overline{F}(x). \tag{1.3}$$

The *pdf* of $X_{U(r)}$ is given by [Ahsanullah, 1995]:

$$f_{X_{U(r)}}(x) = \frac{1}{(r-1)!}[-\log \overline{F}(x)]^{r-1} f(x). \tag{1.4}$$

The joint *pdf* of $X_{U(r)}$ and $X_{U(s)}$ $1 \le r < s$, is given by

$$f_{X_{U(r)}, X_{U(s)}}(x,y) = \frac{1}{(r-1)!(s-r-1)!}[-\log \overline{F}(x)]^{r-1}$$

$$\times [-\log \overline{F}(y) + \log \overline{F}(x)]^{s-r-1} \frac{f(x)}{\overline{F}(x)} f(y),$$

$$\alpha < x < y < \beta, \tag{1.5}$$

where $\overline{F}(x) = 1 - F(x)$.

2. Exact moments of record statistics

Theorem 2.1: For the single moments of Burr distribution,

$$\alpha_r^{j-p} = E[X_{U(r)}^{j-p}] = \frac{(-1)^{r-1}\mu^r}{(r-1)!} \frac{\partial^{r-1}}{\partial \nu^{r-1}} \mathrm{B}\left(\frac{j}{p}, \nu\right), \tag{2.1}$$

where $\nu = \mu - \frac{j}{p} + 1 > 0$ and $\frac{j}{p} > 0$.

Proof: We have,

$$E[X_{U(r)}^{j-p}] = \frac{1}{(r-1)!}\int_0^\infty x^{j-p}[-\log \overline{F}(x)]^{r-1} f(x)dx$$

$$= \frac{\mu p}{(r-1)!}\int_0^\infty x^{j-p}[-\log(1+x^p)^{-\mu}]^{r-1} \frac{x^{p-1}}{[1+x^p]^{\mu+1}} dx.$$

Set $t = \frac{1}{1+x^p}$, to get

$$E[X_{U(r)}^{j-p}] = \frac{(-1)^{r-1}\mu^r}{(r-1)!}\int_0^1 [\log t]^{r-1} t^{\mu-\frac{j}{p}}(1-t)^{\frac{j}{p}-1}\,dt.$$

Note the following results [Prudnikov *et al.,* 1986]

(i) $$\int_0^a x^{\alpha-1}(a^\delta - x^\delta)^{\beta-1}[\log x]^n dx = \frac{a^{\delta(\beta-1)}}{\delta}\frac{\partial^n}{\partial\alpha^n}\left[a^\alpha \mathrm{B}(\beta,\frac{\alpha}{\delta})\right],$$

$$a, \delta, \alpha, \beta > 0$$

(ii) $$\frac{\partial^r B(a,b)}{\partial b^r} = \sum_{k=0}^{r-1}\binom{r-1}{k}[\psi^{(r-k+1)}(b) - \psi^{(r-k-1)}(a+b)]\frac{\partial^k B(a,b)}{\partial b^k},$$

where $B(a,b)$, $a, b > 0$ is the beta function $\psi^{(k)}(x)$ is the k^{th} derivative of $\psi(x) = \frac{d\log\Gamma(x)}{dx} = \frac{\Gamma'(x)}{\Gamma(x)}$, $x \neq 0,-1,-2,\ldots$, which is a diagamma function.

The result given in (2.1) is proved in view of result (i). Using the result (ii) recursively, we can obtain the moments of any value of r.

Theorem 2.2: The product moments of Burr distribution,

$$\alpha_{r,s}^{j\,l-p} = E[X_{U(r)}^{j} X_{U(s)}^{l-p}] = \frac{\mu^s(-1)^{r-1}}{(r-1)!}\sum_{i=0}^{\infty}\frac{\Gamma\left(1-\frac{l}{p}+i\right)}{\Gamma\left(1-\frac{l}{p}\right)i!}$$

$$\times\frac{1}{\left(\mu-\frac{l}{p}+i+1\right)^{s-r}}\frac{\partial^{r-1}}{\partial\eta^{r-1}}\mathrm{B}\left(\frac{j}{p}+1,\eta\right),$$

$$\frac{l}{p}-1 \neq 0,-1,-2,\ldots \qquad (2.2)$$

and

$$\alpha_{r,s}^{j\ l-p} = \frac{\mu^s(-1)^{r-1}}{(r-1)!} \sum_{i=0}^{\frac{l}{p}-1} \frac{(-1)^i \binom{l/p-1}{i}}{\left(\mu - \frac{l}{p} + i + 1\right)^{s-r}}$$

$$\times \frac{\partial^{r-1}}{\partial \eta^{r-1}} \mathrm{B}\left(\frac{j}{p}+1, \eta\right), \quad \frac{l}{p} - 1 \in \mathrm{N}, \tag{2.3}$$

where $\Gamma(.)$ is the gamma function , $\eta = \mu - \frac{j+l}{p} + i + 1 > 0$ and $\frac{j}{p} + 1 > 0$.

Proof: First, we prove (2.2)

$$\alpha_{r,s}^{j\ l-p} = \frac{\mu p}{(r-1)!} \int_0^\infty x^j [-\log(1+x^p)^{-\mu}]^{r-1} \frac{x^{p-1}}{[1+x^p]} I(x)dx, \tag{2.4}$$

where

$$I(x) = \frac{\mu p}{(s-r-1)!} \int_x^\infty y^{l-p} [-\log(1+y^p)^{-\mu} + \log(1+x)^{-\mu}]^{s-r-1}$$

$$\times \frac{y^{p-1}}{[1+y^p]^{\mu+1}} dy.$$

Set $t = \frac{1+x^p}{1+y^p}$, to get

$$I(x) = \frac{\mu(-\mu)^{s-r-1}}{(s-r-1)!(1+x^p)^{\mu - \frac{l}{p} + 1}}$$

$$\times \int_0^1 \left[1 - \frac{t}{(1+x^p)}\right]^{\frac{l}{p}-1} [\log t]^{s-r-1} t^{\mu - \frac{l}{p}} dt. \tag{2.5}$$

Now using the Maclaurin series expansion, we have

$$\left[1-\frac{t}{(1+x^p)}\right]^{\frac{l}{p}-1}=\sum_{i=0}^{\infty}\frac{\Gamma\left(1-\frac{l}{p}+i\right)}{\Gamma\left(1-\frac{l}{p}\right)}\frac{\left(\frac{t}{(1+x^p)}\right)^i}{i!}$$

$$I(x)=\frac{\mu(-\mu)^{s-r-1}}{(s-r-1)!(1+x^p)^{\mu-\frac{l}{p}+i+1}}\sum_{i=0}^{\infty}\frac{\Gamma\left(1-\frac{l}{p}+i\right)}{\Gamma\left(1-\frac{l}{p}\right)i!}$$

$$\times\int_0^1 t^{\mu-\frac{l}{p}+i}[\log t]^{s-r-1}dt. \tag{2.6}$$

Using the result

(iii) $\int_0^1 x^{\alpha-1}[\log x]^n dx=\frac{(-1)^n n!}{\alpha^{n+1}}$ [Prudnikov *et al.*, 1986], equation (2.6) can be written as

$$I(x)=\mu^{s-r}\sum_{i=0}^{\infty}\frac{\Gamma\left(1-\frac{l}{p}+i\right)}{\Gamma\left(1-\frac{l}{p}\right)i!}\frac{1}{(1+x^p)^{\mu-\frac{l}{p}+i+1}\left(\mu+i-\frac{l}{p}+1\right)^{s-r}}.$$

Putting the value of $I(x)$ in (2.4) and setting, $t=\frac{1}{1+x^p}$, we get

$$\alpha_{r,s}^{j\,l-p}=\frac{\mu^s(-1)^{r-1}}{(r-1)!}\sum_{i=0}^{\infty}\frac{\Gamma\left(1-\frac{l}{p}+i\right)}{\Gamma\left(1-\frac{l}{p}\right)i!}\frac{1}{\left(\mu+i-\frac{l}{p}+1\right)^{s-r}}$$

$$\times\int_0^1[\log t]^{r-1}t^{\mu-\frac{j+l}{p}+i}(1-t)^{\frac{j}{p}}dt. \tag{2.7}$$

In view of (i), (2.7) can be written as

$$\alpha_{r,s}^{j\,l-p} = E[X_{U(r)}^{j} X_{U(s)}^{l-p}] = \frac{\mu^{s}(-1)^{r-1}}{(r-1)!} \sum_{i=0}^{\infty} \frac{\Gamma\left(1-\frac{l}{p}+i\right)}{\Gamma\left(1-\frac{l}{p}\right) i!}$$

$$\times \frac{1}{\left(\mu-\frac{l}{p}+i+1\right)^{s-r}} \frac{\partial^{r-1}}{\partial \eta^{r-1}} \mathrm{B}\left(\frac{j}{p}+1, \eta\right), \frac{l}{p}-1 \neq 0,-1,-2,\ldots$$

and hence the Theorem.

Proceeding in the similar way and expanding $\left[1-\frac{t}{(1+x^{p})}\right]^{\frac{l}{p}-1}$ binomially, (2.3) can be proved.

Remark 2.1: Theorem 2.2 reduces to Theorem 2.1, if we put $j = j - p$ and $l = p$ in (2.3).

Remark 2.2: The results obtained in (2.1), (2.2) and in (2.3) can be utilized for obtaining the moments of Lomax distribution $(p=1)$ and log logistic distribution $(\mu=1)$, as they are the special cases of Burr XII distribution.

Table 5.1: Mean of record statistic from Burr distribution $(p=2)$

r	$\mu=2.5$	$\mu=3$	$\mu=3.5$	$\mu=4$	$\mu=4.5$	$\mu=5$	$\mu=5.5$	$\mu=6$
1	0.6667	0.5890	0.5333	0.4909	0.4571	0.4295	0.4063	0.3866
2	1.1340	0.9772	0.8700	0.7912	0.7302	0.6813	0.6410	0.6070
3	1.6198	1.3580	1.1873	1.0658	0.9743	0.9022	0.8438	0.7952
4	2.1753	1.7703	1.5178	1.3439	1.2159	1.1171	1.0383	0.9735
5	2.8362	2.2362	1.8777	1.6384	1.4664	1.3362	1.2337	1.1507

Table 5.2: Covariance of record statistic from Burr distribution $(p = 2)$

r	s	$\mu = 2.5$	$\mu = 3$	$\mu = 3.5$	$\mu = 4$	$\mu = 4.5$	$\mu = 5$	$\mu = 5.5$	$\mu = 6$
1	1	0.22221	0.15380	0.11559	0.09240	0.07676	0.06553	0.05722	0.05054
1	2	0.23016	0.14860	0.10723	0.08290	0.06723	0.05618	0.04816	0.04203
1	3	0.26328	0.16040	0.11111	0.08340	0.06595	0.05421	0.04576	0.03947
1	4	0.31342	0.18110	0.12076	0.08790	0.06811	0.05501	0.04584	0.03894
1	5	0.38060	0.20920	0.13452	0.09540	0.07241	0.05740	0.04715	0.03963
2	2	0.49184	0.29510	0.20310	0.15180	0.11991	0.09833	0.08292	0.07155
2	3	0.56694	0.32070	0.21165	0.15340	0.11827	0.09523	0.07912	0.06731
2	4	0.67761	0.36330	0.23071	0.16240	0.12245	0.09682	0.07925	0.06668
2	5	0.82485	0.42050	0.25760	0.17640	0.13024	0.10115	0.08170	0.06782
3	3	1.00585	0.53080	0.33432	0.23450	0.17614	0.13914	0.11380	0.09566
3	4	1.20644	0.60320	0.36551	0.24890	0.18275	0.14185	0.11428	0.09487
3	5	1.47152	0.69950	0.40871	0.27070	0.19469	0.14838	0.11790	0.09676
4	4	1.98407	0.92850	0.53788	0.35440	0.25419	0.19349	0.15343	0.12589
4	5	2.42421	1.07850	0.60243	0.38610	0.27110	0.20273	0.15855	0.12849
5	5	3.81607	1.59320	0.85244	0.52960	0.36307	0.26637	0.20538	0.16419

3. Application of the moments

The exact and explicit expressions for single moments of record statistics given in (2.1) allow us to evaluate the means of all record statistics. Table 5.1 presents the means of $X_{U(r)}, r = 1, \ldots, 5$ for Burr XII distribution at $(p = 2)$. For the computation of variances and covariances, the product moments $\alpha_{r,s}^{11}$ $1 \le r < s$ were computed first and then variances and covariances are computed. For $r \ge s$ the values of $\alpha_{r,s}$ were filled in by using the symmetry of the variance–covariance matrix $(\alpha_{r,s})$. Table 5.2 provides the variances and covariances of record statistics for

$(p=2, \mu=2.5:0.5:6)$, MATLAB has used for computation of the moments as beta and polygamma functions are available there.

Assume $Y_1, Y_2, \ldots,$ to be an infinite sequence of *iid r.v's* with *pdf*

$$g(y)=\mu p\left(\frac{y-\theta}{\sigma}\right)^{p-1}\left[1+\left(\frac{y-\theta}{\sigma}\right)^{p}\right]^{-(\mu+1)},$$

$$y>\theta;\ \mu, p, \sigma>0. \tag{3.1}$$

Let $Y_{U(1)} \leq Y_{U(2)} \ldots \leq Y_{U(s)}$ be the first s observed record values from the above sequence. Then $\underline{X}=[X_{U(1)}, X_{U(2)}, \ldots, X_{U(s)}]$ where $X_{U(i)}=\left(\frac{Y_{U(i)}-\theta}{\sigma}\right)$, $i=1,2,\ldots,s$ is the vector of s observed record statistics from a population with the standard Burr XII distribution with *pdf* and *df* given in (1.1) and (1.2) respectively. Then we can write the best linear unbiased estimators (BLUE's) of θ and σ [Arnold *et al.*, 1998]

$$\hat{\theta}=[a_1 Y_{U(1)}+a_2 Y_{U(2)}+\ldots+a_s Y_{U(s)}] \tag{3.2}$$

$$\hat{\sigma}=[b_1 Y_{U(1)}+b_2 Y_{U(2)}+\ldots+b_s Y_{U(s)}], \tag{3.3}$$

where $a_i{}'s$ and $b_i{}'s$ are the entries of the matrix $C=(A'V^{-1}A)^{-1}A'V^{-1}$ with $A=(\underline{1}\ \ \underline{\alpha}), \underline{1}'==(1,\ldots,1)_{1\times s}$, $\underline{\alpha}'==(\alpha_1,\ldots,\alpha_s)_{1\times s}$ where $\underline{\alpha}$ is the mean of the first s record values and V^{-1} is the inverse of the covariance matrix $V=(\sigma_{r,s})_{s\times s}$. Variances and covariances of these estimators are given by

$$Var(\hat{\theta})=d_{11}\sigma^2,\ Var(\hat{\sigma})=d_{22}\sigma^2 \text{and } Co\text{var}(\hat{\theta},\hat{\sigma})=d_{12}\sigma^2, \tag{3.4}$$

where

$$D=\begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}\sigma^2=(A'V^{-1}A)^{-1}.$$

The necessary coefficients in (3.2) and (3.3) required for the BLUE of θ and σ based on of record statistics from the Burr XII distribution at $(p=2)$ are computed in Table 5.3 and in Table 5.4 respectively. The variances and covariances of the BLUE's are presented in Table 5.5. Here coefficients are computed for Burr XII distribution $(p=2, \mu=2.5:0.5:6)$. Similarly we can obtain BLUE for any other choice of the parameters μ, p.

Table 5.3: Coefficients for the BLUE of θ for $p=2$

s	$\mu=2.5$	$\mu=3$	$\mu=3.5$	$\mu=4$	$\mu=4.5$	$\mu=5$	$\mu=5.5$	$\mu=6$
2	2.4267	2.5173	2.5839	2.6347	2.6737	2.7057	2.7311	1.8333
	-1.4267	-1.5173	-1.5839	-1.6347	-1.6737	-1.7057	-1.7311	-0.8333
3	1.7828	1.8039	1.8220	1.8383	1.8510	1.8635	1.8715	1.2156
	-0.1634	-0.0767	-0.0135	0.0326	0.0695	0.0965	0.1233	0.2992
	-0.6194	-0.7272	-0.8085	-0.8709	-0.9205	-0.9600	-0.9949	-0.5148
4	1.5777	1.5648	1.5591	1.5583	1.5602	1.5632	1.5635	1.0126
	-0.1272	-0.0460	0.0106	0.0510	0.0792	0.1008	0.1221	0.2276
	-0.1302	-0.1013	-0.0725	-0.0484	-0.0267	-0.0078	0.0086	0.1295
	-0.3202	-0.4176	-0.4971	-0.5608	-0.6127	-0.6562	-0.6942	-0.3698
5	1.4868	2.2973	1.4293	1.4168	1.4095	1.4063	1.4009	0.9146
	-0.1128	-0.0522	0.0214	0.0587	0.0825	0.1028	0.1248	0.1884
	-0.1188	-0.1381	-0.0586	-0.036	-0.0121	0.0028	0.0158	0.1068
	-0.0801	-0.1207	-0.0693	-0.056	-0.0437	-0.0339	-0.0283	0.0756
	-0.1751	-0.4012	-0.3228	-0.3835	-0.4362	-0.4780	-0.5132	-0.2853
	2.4267	2.5173	2.5839	2.6347	2.6737	2.7057	2.7311	1.8333

Table 5.4: Coefficients for the BLUE of σ for $p=2$

s	$\mu=2.5$	$\mu=3$	$\mu=3.5$	$\mu=4$	$\mu=4.5$	$\mu=5$	$\mu=5.5$	$\mu=6$
2	-2.1400	-2.5760	-2.9700	-3.3300	-3.6617	-3.9714	-4.2608	-4.1667
	2.1400	2.5760	2.9700	3.3300	3.6617	3.9714	4.2608	4.1667
3	-1.2443	-1.4542	-1.6467	-1.8273	-1.9915	-2.1527	-2.3005	-1.5808
	0.3829	0.3105	0.2424	0.1840	0.1229	0.0796	0.0318	-0.5741
	0.8615	1.1436	1.4042	1.6433	1.8686	2.0731	2.2686	2.1549
4	-0.9782	-1.1043	-1.2223	-1.3347	-1.4429	-1.5484	-1.6450	-0.8362
	0.3359	0.2656	0.2036	0.1517	0.1046	0.0709	0.0344	-0.3115
	0.2266	0.2277	0.2161	0.1965	0.1823	0.1572	0.1334	-0.2087
	0.4157	0.6110	0.8026	0.9865	1.1560	1.3203	1.4771	1.3564
5	-0.8671	-1.5786	-1.0252	-1.1014	-1.1766	-1.2517	-1.3200	-0.5144
	0.3182	0.3335	0.1871	0.1389	0.0989	0.0672	0.0291	-0.1830
	0.2126	0.2941	0.1949	0.1760	0.1565	0.1371	0.1189	-0.1340
	0.1222	0.2038	0.1531	0.1542	0.1508	0.1436	0.1461	-0.1051
	0.2140	0.5164	0.4900	0.6323	0.7705	0.9039	1.0259	0.9365
	-2.1400	-2.5760	-2.9700	-3.3300	-3.6617	-3.9714	-4.2608	-4.1667

Table 5.5: Variances and covariances of the BLUE of θ and σ in terms of $\sigma^2\,(p=2)$

s	$\mu=2.5$	$\mu=3$	$\mu=3.5$	$\mu=4$	$\mu=4.5$	$\mu=5$	$\mu=5.5$	$\mu=6$
2	0.4039	0.3330	0.2835	0.2468	0.2193	0.0911	0.4039	0.3330
	0.9200	0.8694	0.8353	0.8107	0.7933	0.6615	0.9200	0.8694
	-0.5156	-0.4584	-0.4170	-0.3851	-0.3607	-0.1677	-0.5156	-0.4584
3	0.2743	0.2245	0.1905	0.1652	0.1465	0.0668	0.2743	0.2245
	0.5293	0.4831	0.4523	0.4304	0.4148	0.2348	0.5293	0.4831
	-0.2906	-0.2537	-0.2283	-0.2090	-0.1947	-0.0658	-0.2906	-0.2537
4	0.2305	0.1870	0.1578	0.1364	0.1206	0.0586	0.2305	0.1870
	0.4149	0.3672	0.3357	0.3138	0.2973	0.1241	0.4149	0.3672
	-0.2198	-0.1878	-0.1665	-0.1511	-0.1395	-0.0356	-0.2198	-0.1878
5	0.2093	0.1684	0.1410	0.1215	0.1072	0.0543	0.2093	0.1684
	0.3662	0.3165	0.2835	0.2605	0.2438	0.0780	0.3662	0.3165
	-0.1877	-0.1571	-0.1370	-0.1229	-0.1127	-0.0216	-0.1877	-0.1571

Example: Let us consider the case where the components have failure times which follow a Burr XII distribution with $(\mu, p, \theta, \sigma) = (3,2,10,4)$. Suppose that we observe the following simulated observed failure times 10.9105 , 12.0759, 11.3485, 11.2747, 12.6274, 10.7958, 11.7932, 11.7818, 10.9605, 11.3118, 10.3771, 10.3156, 11.2370 and 11.8683.

We get the record statistics from the observed data as follows:

10.9105, 12.0759, 12.6274

Here then, for the recorded data analysis with $s = 3, \mu = 3$ and $p = 2$; $\underline{\alpha}$ and ν are obtained From Table 5.1 and Table 5.2 respectively. The coefficients in (3.2) and (3.3) are presented in Table 5.3 and Table 5.4, respectively. Therefore, the BLUE's of θ and σ are computed to be $\hat{\theta} = 9.5726$ and $\hat{\sigma} = 2.3242$. The corresponding variances and covariances of $\hat{\theta}$ and $\hat{\sigma}$ (in the Table 5.5) are computed to be $V(\hat{\theta}) = 0.2245\sigma^2 = 3.5920$, $V(\hat{\sigma}) = 0.4831\sigma^2 = 7.7296$ and $Cov(\hat{\theta}, \hat{\sigma}) = -0.2537\sigma^2 = -4.0592$. Let us consider the true population mean $\tau = E(Y) = \theta + \alpha_1\sigma = 12.536$. Now suppose $\tau^* = \left[\dfrac{Y_{U(1)} + Y_{U(2)} + Y_{U(3)}}{3}\right]$ is the mean of observed records. We would have $\tau^* = 11.871267$ and $S.E(\tau^*) = 1.9951$. The BLUE of $\hat{\tau} = \hat{\theta} + \alpha_1\hat{\sigma} = 10.9415$. The standard error of $\hat{\tau}$ is computed to be $S.E(\hat{\tau}) = 1.2214$. Therefore, the BLUE performs better than the mean of the observed records in the sense of standard error.

In the context of prediction of the future record observations, suppose we observe only the first r recorded observations $\underline{Y} = [Y_{U(1)}, Y_{U(2)}, \ldots, Y_{U(r)}]$ and the goal is to predict $Y_{U(s)}$, where $1 \le r < s$. When F belongs to a location and scale parameter family, the most well-known predictor is the best linear unbiased predictor (BLUP) (see, for the example in the context of order statistics, Kaminsky and Nelson,1975) of $Y_{U(s)}$ is given by

$$\hat{Y}_{U(s)} = (\hat{\theta} + \hat{\sigma}\alpha_s) + \underline{w}'V^{-1}(\underline{X} - \hat{\theta}.\underline{1} - \hat{\sigma}\underline{\alpha}),$$

where $\underline{\alpha}$ is the mean of the first r record value and $\underline{w}'$ is the vector of the covariances between the s^{th} future record statistics and the first r recorded observations. The mean square prediction error (MSPE) of $Y_{U(s)}$ is found to be (Raqab, 1996)

$$MSPE[Y_{U(s)}] = [\underline{E}'\,V\underline{E} + \sigma_{n,n} - 2\underline{E}'\,\underline{w}\,]\sigma^2,$$

where

$$\underline{E}' = (1.\alpha_n)(A'V^{-1}A)^{-1}A'V^{-1} + \underline{w}'V^{-1}[I - A(A'V^{-1}A)^{-1}A'V^{-1}].$$

In our data set up, we have observed three record statistics. Table 5.1, 5.2, 5.3, 5.4 and 5.5 are used to compute the BLUP of the future record statistics $Y_{U(4)}$ and $Y_{U(5)}$ based on the first three observed record statistics. Their values are computed to be $Y_{U(4)} = 13.5697$ and $Y_{U(5)} = 14.6322$ and the corresponding MSPE's are given by

$$MSPE[Y_{U(4)}] = 0.3014\sigma^2 = 4.8263$$

and

$$MSPE[Y_{U(5)}] = 0.3796\sigma^2 = 6.0738.$$

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